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ABSTRACT

This volume contains the plenary or reactor papers presented at a conference on reform in algebra held in Leesburg, Virginia, December 9-12, 1993. Papers included are: (1) "Introduction" (C. B. Lacampagne); (2) "Summary" (C. B. Lacampagne); (3) "Recommendations" (C. B. Lacampagne); (4) "The Development of Algebra and Algebra Education" (V. J. Katz); (5) "Long-Term Algebra Reform: Democratizing Access to Big Ideas" (J. J. Kaput); (6) "Algebra in the K-12 Curriculum" (G. Burrill); (7) "What Is the Appropriate K-12 Algebra Experience for Various Students?" (J. Fey); (8) "Algebra at the College Level" (M. Artin); (9) "Algebra Initiative" (V. Pless); (10) "Algebra and the Technical Workforce" (H. Pollak); (11) "Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce" (S. Garfunkel); (12) "A Cognitive Perspective in the Mathematical Preparation of Teachers: The Case of Algebra" (A. G. Thompson & P. W. Thompson); (13) "Preparing Teachers to Teach Algebra for All: Preliminary Musings and Questions" (M. Enneking); and (14) "Algebra for All: Dumbing Down or Summing Up?" (L. A. Steen). Appendices include the conference agenda; Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment; and a participant list. (MKR)

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The Algebra Initiative Colloquium

Volume 1

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The Algebra Initiative Colloquium

Volume 1

Papers presented at a conference on reform in algebra
December 9–12, 1993

Edited by

Carole B. Lacampagne
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U.S. Department of Education
Office of Educational Research and Improvement
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Introduction

Carole B. Lacampagne

This publication contains plenary and reactor papers presented at the Algebra Initiative Colloquium held on December 9-12, 1993, in Leesburg, Virginia (see the agenda, appendix A). It also contains a summary and recommendations from the working groups.

The Algebra Initiative Colloquium, sponsored by the U.S. Department of Education's Office of Educational Research and Improvement, addressed specific issues in the algebra curriculum and its teaching and learning. Algebra was chosen from among the many subject areas of mathematics because it is the language of mathematics; algebra is central to the continued learning of mathematics at all levels. Moreover, although several groups are already at work on the reform of algebra, there has been little dialog bridging educational levels. The Algebra Initiative Colloquium fostered such dialog (see the Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment, appendix B).

Fifty-one distinguished algebra teachers, mathematics education researchers, algebraists, and mathematics experts from federal agencies attended the Colloquium (see Participant List, appendix C). In addition to hearing and discussing the plenary and reactor addresses, participants were assigned to one of four working groups to debate the issues and, where possible, to come up with a set of recommendations. The working groups and their foci are listed below:

- Working Group 1: Creating an appropriate algebra experience for *all* grades K-12 students;
- Working Group 2: Educating teachers, including K-8 teachers, to provide these algebra experiences;
- Working Group 3: Reshaping algebra to serve the evolving needs of the technical work force; and
- Working Group 4: Renewing algebra at the college level to serve the future mathematician, scientist, and engineer.

Prior to the Colloquium, working group participants wrote short papers on their group's focus area and shared papers within the working group, thus getting a head start on Colloquium deliberations. These papers, revised after the Colloquium, together with recommendations from each working group, appear in the companion publication, *The Algebra Initiative Colloquium, Volume 2*.

A short document for teachers and policymakers was prepared on the basis of recommendations from the Colloquium. This document, *Algebra for All: A Lever, not a Wedge*, will soon be available through the National Institute on Student Achievement, Curriculum, and Assessment.

Summary

Carole B. Lacampagne

Sponsored by the U.S. Department of Education, The Algebra Initiative Colloquium generated a number of questions, concerns, and recommendations to promote the reform in algebra across the educational gamut, K through 16.

According to Lynn Steen, "The major theme of this conference is very simple: Algebra is broken but nonetheless essential." From this premise, several subthemes/questions/concerns emerged that might be considered in setting an algebra agenda. These include:

- Algebra, the new civil right;
- The chasm that separates K-12 and 13-16 algebra education;
- A new algebra curriculum and pedagogy for preservice teachers and support and practice for current teachers;
- Algebra for the technical work force;
- The story line of algebra; and
- The brick wall.

Algebra, The New Civil Right

As Robert Moses so eloquently put it, "Algebra is the new civil right." Working Group 1 (Creating an appropriate algebra experience for all grades K-12 students) calls algebra "... the academic passport for passage into virtually every avenue of the job market and every street of schooling." Moreover, since students from non-Asian minority groups are less likely to have obtained this passport than others, algebra becomes an equity issue.

Working Group 1 recommended that all students have significant experiences in algebra before the end of grade 8 and that these experiences be the equivalent of current ninth-grade algebra. They also recommended that these experiences be strands that flow through K-8 mathematics.

Algebra for all is a bold statement. As keynote speaker Victor Katz pointed out, algebra has a history of being an elitist subject meant for the education of future priests or leaders. Many of the story problems found in current high school algebra texts date back to antiquity and are artificial, not real-life problems. If we are to implement algebra for all, we must overcome this longstanding elitist tradition.

A question that plagued Colloquium participants was, "How do we insure that 'algebra for all' is not 'dumbing down' algebra?" The mathematical community as well as parents of college-bound students will and should demand sound preparation in algebra for the college bound. We will be faced with building an algebra curriculum and pedagogy that will support the needs of all students.

Integrating algebra into the K-8 mathematics curriculum was also a topic of considerable interest to Colloquium participants. Such integration would eliminate algebra from its current gate-keeper function, for algebra would be learned over a period of 8 years. The college death knoll—failing algebra—would no longer exist. Moreover, many countries already integrate algebra into their K-8 curriculum through the use of a strands approach; that is, each year a strand of algebra as well as strands of arithmetic and geometry are taught. Several curriculum projects in the United States have developed or are developing such an approach. These curriculum projects need to be tested in classrooms to prove their effectiveness, and promising practices need to be disseminated.

Several participants were concerned that we do not know enough about how children and adults develop algebraic concepts. More research is needed on how algebraic concepts are developed. Findings from this research should then be considered when designing algebra curriculum.

Looking to the future when algebra could be folded into the K-8 curriculum raised several questions:

- What do we do with ninth-grade mathematics once algebra has been learned by eighth grade? How do we prepare for such change?
- How do we influence public policy—state legislators, boards of education? State legislators and boards of education must adjust their ninth-grade mathematics requirements to the new curriculum. How do we get parents to buy into this new alignment?

The Chasm That Separates K-12 and 13-16 Algebra Education

It became disturbingly apparent as the Algebra Initiative Colloquium progressed that mathematicians involved with research in the learning and teaching of algebra at the K-12 level seldom talk with those involved in algebra at the 13-16 level.

Working Group 4 (Renewing algebra at the college level to serve the future mathematician, scientist, and engineer) suggested several areas in which further dialog was needed between K-12 and 13-16 people to:

- Lay the basis in high school for some important ideas from linear and abstract algebra;
- Find new ways to engage students in linear and abstract algebra;
- Develop places across the entire algebra experience to expose students to the use of proof; and
- Search for the "big themes" which run throughout the algebra experience.

Initiatives for getting this dialog going are already under way, thanks to the groundwork laid at the Colloquium.

A New Algebra Curriculum and Pedagogy for Preservice Teachers and Support and Practice for Current Teachers

Reform in the school algebra experience necessitates a change in the algebra experiences of pre- and in-service mathematics teachers. Questions wrestled with by Working Group 2 (Educating teachers, including K-8 teachers) to provide these algebra experiences included:

- How do we help elementary school teachers gain the knowledge, pedagogical skills, and desire to integrate algebra into the K-8 curriculum? How can we prepare parents and communities for this new approach?
- How can we encourage mathematics and mathematics education faculty at the college level to model the pedagogy we wish to see prospective teachers use in the schools?
- What experiences with mathematical modeling should pre- and in-service teachers have in order to teach algebra effectively?

Working Group 2 believed that all teachers of pre-college algebra (K-12) need an in-depth understanding of numeracy and quantitative reasoning. They should also have a working knowledge of how to use technology in instruction and how to access real-world examples that employ algebra. They should possess a belief in the value of mathematics and a commitment to the development of algebraic thinking for all students.

Algebra for the Technical Work Force

Working Group 3 had several recommendations for reforming the K through 14 mathematics curriculum and teaching, including that all students should study the same mathematics through grade 11, with algebra playing a significant part in the curriculum. Moreover, they saw mathematical modeling as a central or organizing theme for school mathematics. These recommendations led to such further concerns as:

- What should be the common mathematics curriculum required of students through grade 11?
- What alternative mathematics courses should be taught in grade 12?
- How do we mount a serious study of what algebra/mathematics is used in technical jobs?
- How do we remediate the remedial mathematics programs in our colleges?
- How do adult learners best learn algebra?
- Currently, we teach content, then application. If mathematical modeling were our focus, we would teach applications then content on a need-to-know basis. Can or should we embark on such a radical change?
- How can we begin to work with the vocational/technical education community to provide a meaningful algebra experience in all its programs?

The Story Line of Algebra

At the Colloquium, participants had trouble identifying the "story line" of algebra. Specifically, Bob Moses challenged us to describe what (high school, linear, or abstract) algebra is and why a student should study it. If algebraists and teachers of algebra cannot explain to each other what algebra is all about, how can we expect to engage prospective students, their parents, and the community in the study of and support for algebra? Participants felt that the mathematical community must be able to answer such questions and to communicate these answers to the public at large.

The Brick Wall

Working Group 4 spent much time deliberating on how to confront the "brick wall"; that is, how to help students through their first proof course (usually abstract algebra). They discussed the possibility of starting a dialog, perhaps in a journal like *The American Mathematical Monthly,* on how to help students over (around) the brick wall and on sharing concrete examples and problems in abstract algebra among those who teach abstract algebra.

The big problem that this group wrestled with was how to reconcile the need for formal proof at the college level with the trend to downplay formal proof in the schools in favor of communicating ideas and understanding.

Participants at the Algebra Initiative Colloquium made a good start in identifying problems in the learning and teaching of algebra at all levels. They made several recommendations to help solve these problems. But many questions they raised remain unanswered. It is hoped that federal agencies working with leaders in the mathematical community will provide leadership in renewing algebra to meet the needs of the 21st century.

Recommendations

Carole B. Lacampagne

Participants in the Algebra Initiative Colloquium were divided into four working groups. They prepared and exchanged short papers before the Colloquium, and came up with recommendations during the Colloquium. Each working group's recommendations are summarized below.

Working Group 1 Creating an appropriate algebra experience for ALL K-12 students

(1) All students should have significant experiences in algebra (totaling at least 1 year of work) before the end of eighth grade.

(2) All students need to learn the following aspects of algebra:

- The representation of phenomena with symbols and the use of these symbols sensibly;
- The use of variables to describe patterns and give formulas involving geometric, physical, economic, and other relations;
- Simple manipulations with these variables to enable other patterns to be seen and variations to be described;
- The solving of simple equations, inequalities, and systems by hand, and of more complicated equations, inequalities, and systems by machine; and
- The picturing and examination of relationships among variables using graphs, spreadsheets, or other technology.

(3) All students should be introduced to operations and their properties in the various number systems (such as whole numbers, integers, real numbers, and complex numbers), and on objects other than numbers (such as sets, matrices, transformations, and propositions).

(4) The mastery of a small number of paper-and-pencil manipulative skills should be expected of all students. Symbol manipulation software and calculators should be used for other manipulative skills, perhaps including skills not now accessible or easily accessible with paper-and-pencil techniques.

(5) Both short-term and long-term strategies are needed to ensure that all students study a significant amount of algebra.

(6) A much expanded research base is required to make continued progress on recommendations 1-5, including inquiry on:

- What constitutes symbol sense and how it is developed;
- The roles and balance of "rote skills" and machine-performed computations;
- The development of modeling skills and the character of mathematical abstraction;
- The successes and difficulties of various approaches to teaching algebra (including attempts in other countries);
- The use of mathematical and algebraic thinking, often unrecognized, in work settings; and
- The creation of effective instructional environments, and the kind of teacher understanding required to foster algebraic thinking in such environments.

Working Group 2 Educating teachers, including K-8 teachers, to provide appropriate algebra experiences for their students

(1) Crucial concepts for teachers of pre-college algebra include:

- A working knowledge of the use of technology in instruction;
- Conceptualization of relations among quantities;
- A commitment to the development of algebraic thinking for *all* students;
- An understanding of the role of algebra as a gateway to academic development and full participation in citizenship; and
- Access to real-world examples using algebra.

(2) Model teacher preparation curricula need to be developed in line with the ideas of a strands approach for the teaching of algebra for all and in light of new technologies.

(3) Elementary and middle school teacher preparation programs should pay particular attention to the mathematics of quantity and change where opportunities to learn algebra occur. Applications through technology and skills in using symbolic manipulators and graphing utilities should be an integral part of such programs.

(4) Pedagogy should be content-specific. There should be a pairing of content and pedagogy work such as shadow seminars (seminars on teaching issues associated with specific content courses).

(5) Reform in algebra teaching should include attention to effective development such as:

- Studies both to synthesize existing knowledge and to create new knowledge concerning the ways in which the teaching of mathematics encodes biases which are often not recognized by teachers;
- Mathematics examples which can be included in multicultural education courses often required of prospective teachers; and
- The development of a syllabus for a course on the multicultural history of mathematics which should be required of all mathematics education students.

(6) Establish a strong link between teacher preparation programs and "the world of practice"—the schools. The school should be the focus of change for practicing teachers, with teachers determining the inservice training that they need.

(7) Faculty enhancement should provide practicing teachers with opportunities to:

- Develop and experience a broad repertoire of strategies to enhance the conceptual understanding of their students;
- Become more knowledgeable about how students learn and to asses the learning of their students;
- Explore curricular issues related to the learning of algebra and preparing K-12 teachers to implement new curriculum; and
- Develop or become familiar with exemplary curriculum materials which enhance understanding of algebra.

(8) Model programs need to be identified or developed that are characterized by:

- A team approach to the identification of needs;
- Support by school district administrators;

- A variety of sources for professional development, including universities, professional organizations, school district administrators or staff, and other practicing teachers; and
- Mutual respect and parity among all players.

Working Group 3 Reshaping algebra to serve the evolving needs of the technical workforce

(1) All students should study the same mathematics curriculum through grade 11, but not the curriculum that currently exists. Algebra should be a significant part of that curriculum, although not necessarily a discrete course.

(2) Mathematics courses in grade 12 should offer many alternatives.

(3) All mathematics courses (K-16) should integrate preparation for the technical work force into the curriculum; pedagogy in those courses should prepare students to become independent learners of mathematics and other technical subjects.

(4) The mathematics curriculum should be grounded in problem solving that reflects real-world situations and offers a variety of methods of solution.

(5) Students should have opportunities to participate in group work, make appropriate use of technology, and develop their communication skills, including reading and writing technical materials.

(6) Community college mathematics faculty must be involved in curriculum development in technical areas—health, human services, business and information management, agriculture and agribusiness, and engineering and industry. Community colleges should take an active role in needs assessment and articulation with secondary schools and 4-year colleges and universities.

(7) Research is needed to determine what mathematics curriculum and pedagogy best suit the needs of the adult learner. Specifically, research is needed on whether the several semesters of remedial arithmetic and algebra that many community college students now take before beginning a technical mathematics course can be circumvented.

(8) Research is needed on the kinds of mathematics students will need to be successful employees in a technical work force. Discovering how

mathematics is used in the workplace is a subtle research problem that should be carried out by mathematicians who can recognize when mathematics is being used in situations that do not look like textbook examples.

Working Group 4 Renewing algebra at the college level to serve the future mathematician, scientist, and engineer

(1) There should be more dialog between K-12 and 13-16 educators to:

- Lay the basis in high school for some important ideas from linear and abstract algebra—including various uses of proof and explanation, reasoning about binary operations, and working with symbol systems—important for all students, not just the college bound;
- Introduce topics from advanced algebra, especially from linear algebra;
- Find new ways to connect with and engage students in linear and abstract algebra;
- Build on and be informed by the various reforms and changes taking place at the secondary level to plan the algebra experiences of university students (i.e., mixing deduction with experiment, using cooperative learning, developing project-based activities, incorporating new technologies); and
- Search for algebraic themes running throughout the K-16 mathematics experience.

(2) Linear and abstract algebra courses should:

- Integrate technology;
- Use applications from practical situations or from other parts of mathematics; and
- Find ways to confront "the brick wall" which many students encounter in their first proof course (usually abstract algebra), including:
  - Generalization only after giving concrete examples, and
  - Develop proof techniques through the linear algebra course.

(3) Students should use proof as a research technique throughout their mathematics experience. Activities should be developed in lower level courses that illustrate the value of proof as a mechanism for communication and for establishing and discovering new results.

The Development of Algebra and Algebra Education

Victor J. Katz
University of the District of Columbia

The goal of this conference is to begin to reform the learning and teaching of algebra at all levels, kindergarten through graduate school. That is a major challenge, inasmuch as algebra learning and teaching have been going on in schools around the world for at least 4,000 years. As a first step, then, it may be well to take a glimpse at what the learning and teaching of algebra has been over that time period.

We will begin this task the way one begins any mathematical discussion, with a definition. What is algebra? Let us take a look at a few algebra books to see how various authors have defined this subject. First, we consider the earliest true text in algebra, *Al-kitab al-muhtasar fi hisab al-jabr w'al-muqabala* (*The Condensed Book on the Calculation of al-Jabr and al-Muqabala*), by Muhammad ibn Musa al-Khwarizmi, dating to about 825. Al-Khwarizmi simply says that he was composing "a short work on calculating by the rules of al-jabr and al-muqabala confining it to what is easiest and most useful in arithmetic." (Rosen, p. 3) That is not a useful definition unless we know better what al-jabr and al-muqabala are. So, let us look at another Islamic work with virtually the same title, the *Al-jabr w'al Muqabala* of Omar Khayyam of about 1100. Omar says, "One of the branches of knowledge needed in that division of philosophy known as mathematics is the science of al-jabr and al-muqabala which aims at the determination of numerical and geometrical unknowns." (Kasir, p. 43) That is somewhat clearer—the subject is about solving equations—and if we look further at the text we can figure out what al-jabr and al-muqabala mean.

Al-jabr means the operation of transposing a subtracted quantity on one side of an equation to the other side where it becomes an added quantity, while al-muqabala refers to the reduction of a positive term by subtracting equal amounts from both sides of an equation. For example, converting $3x + 2 = 4 - 2x$ to $5x + 2 = 4$ is an example of al-jabr, while converting the latter to $5x = 2$ is an example of al-muqabala. But maybe we would like a more modern version of a definition. Consider *A Treatise of Algebra in Three*

Parts, by Colin Maclaurin, published in 1748. Maclaurin defined algebra as "a general method of computation by certain signs and symbols which have been contrived for this purpose and found convenient." (Maclaurin, p. 1) Leonhard Euler, in his *Vollständige Anleitung zur Algebra* (*Complete Introduction to Algebra*) of 1767, writes, "Algebra has been defined as the science which teaches how to determine unknown quantities by means of those that are known." (Euler, p. 186) In all these definitions, there seems to be one central idea: algebra deals with the solving of equations, and with the manipulation of signs representing numbers which is necessary for this goal. (The Islamic authors did not speak of symbols, because they did not use any.) Nineteenth century algebra texts seemed to agree with this definition.

Let us look at two more definitions. First, from *The Normal Elementary Algebra*, by Edward Brooks (1871): "Algebra is a method of investigating quantity by means of general characters called symbols." (Brooks, p. 19) Second, from *Elements of Algebra* by G. A. Wentworth (1881): "The science which employs letters in reasoning about numbers, either to discover their general properties or to find the value of an unknown number from its relations to known numbers, is called Algebra." (Wentworth, p. 4) So, we now have a definition. Interestingly, however, if you look through 20th century algebra texts, both elementary and advanced, there is usually no definition of the subject. Why not?

The next step after the definitions in a mathematical presentation is a discussion of the theorems to be proved. Although these will perhaps not be proved with the normal rigor of pure mathematics, the basic principles which will be illustrated in the remainder of this article are the following: First, algebra has always been taught through problem solving—and the problems have not changed much over the years. Second, the problems are not real-life problems; they are usually quite artificial. Third, up until the 18th century, algebra was taught through the use of ideas from other fields of mathematics, particularly geometry. And fourth, abstraction in algebra came about only after the consideration of a large number of concrete examples.

Where does algebra begin? Because we have defined algebra as a science concerned with the solving of equations, we may as well begin with the earliest records we have of such a task, namely, the *Rhind Mathematical Papyrus*—or perhaps it should be called the A'h-mose papyrus after the Egyptian scribe who wrote it around 1650 B.C. rather than after the Scotsman who purchased it in Luxor in 1858. In any case, the papyrus is a copy of an original which dates back perhaps 200 years earlier. Problem 24 reads (in translation): "A quantity and its 1/7 added together become 19. What is the quantity?" (Chace, p. 67) The method of solution, developed in detail on the papyrus, is what is now known as false position. We make a convenient guess, here 7, and note that 7 plus 1/7 of itself is 8, not 19. But since 19 divided by 8 is 2 3/8, the correct answer must be 7 multiplied by this same proportionality factor, 2 3/8. Thus, the answer is 16 5/8. The scribe then checks that the result is correct. As is typical in a problem solving text, the scribe follows this example with several more problems of the same type, here written in notation developed in the 17th century:

24. $x + (1/7)x = 19$
25. $x + (1/2)x = 16$
26. $x + (1/4)x = 15$
27. $x + (1/5)x = 21$
28. $x + (2/3)x - (1/3)[x + (2/3)x] = 10.$

We note that, although some of the problems of the papyrus are stated in terms of real objects, this problem—and many others—are stated purely in terms of numbers. They are evidently designed to teach a method of solving problems. Furthermore, the method is to be taught only to a limited few; for, as Ah-mose states in the introduction, the papyrus provides "accurate reckoning—the entrance into the knowledge of all existing things and all obscure secrets." (Chace, p. 27) Such knowledge, in ancient Egypt, could only be shared with members of the priestly class. In fact, throughout most of recorded history, algebra was only taught to the "few," not the "many." So the concept of "algebra for all" is a very recent one and one that will take a lot of work to make into a reality.

Let us move to ancient Babylonia in the same time period—around 1700 B.C. Here we have much more evidence of algebra—equation solving—because the Babylonian scribes wrote on clay tablets, which were then baked, rather than on papyrus. Thousands of the clay tablets have survived the centuries, but only a very few of the papyrus scrolls.

We want to consider one particular tablet, BM (British Museum) 13901. This tablet contains 24 problems, some of which are given here in modern notation [Gandz, pp. 497-501]:

$$x^2 + x = 3/4$$
$$x^2 - x = 870$$

$$(2/3)x^2 + x = 286\ 2/3$$

$$x^2 + (4/3)x = 11/12$$

$$11x^2 + 7x = 6\ 1/4$$

$$x^2 - (1/3)x = 1/12$$

$$x^2 + y^2 = 1300,\ xy = 600.$$

We look at the fourth one in detail. A direct translation of the cuneiform gives the problem as follows: "The sum of the area of a square and 4/3 of the side is 11/12." For the solution, the scribe tells us to take half of 4/3, giving 2/3, square the 2/3, giving 4/9, then add this result to 11/12, giving 1 13/36. The square root of this, namely 7/6, is then found. Subtracting 2/3 from 7/6 gives the value 1/2 for the side. We can easily translate this Babylonian procedure into a modern quadratic formula for solving $x^2 + bx = c$, namely

$$x = \sqrt{\left(\frac{b}{2}\right)^2 + c} - \frac{b}{2}.$$

There are two major points to notice about this and other Babylonian tablets containing quadratic problems. First, the scribe does not indicate how the algorithm was derived, but just presents it numerically. And in fact, given that the problem asks us to add an area and a side, it would seem that it is a purely numerical problem having really nothing to do with the geometrical meaning of the words used to designate the unknowns. Nevertheless, a close reading of the wording of the tablets seems to indicate that the scribe had in mind a geometric procedure. It appears that the multiple 4/3 x is to be considered as a rectangle with length x and width 4/3, a rectangle which is then added to the square of side x. Under this interpretation, the scribe's procedure amounts to cutting half of the rectangle off from one side of the square and moving it to the bottom. Adding a square of side b/2 "completes the square." It is then evident that the unknown length x is equal to the difference of the side of the new square and b/2, exactly as the formula implies (see figure 1). (Høyrup, pp. 38-65)

A second point is that this tablet, like many other Babylonian tablets, consists of a long set of problems, all of the same general type. The problems are usually stated in some concrete form, as here in terms of the side and area of a square. But can we consider them real-life problems? Would a quadratic equation like this one come out of a real situation? It would seem that the quadratic problems here are just as artificial as the majority of problems in current algebra texts. After all, most of the problems of this set—and of others—have the same answer. It thus appears that the tablets were used to develop techniques of solution. In other words, the purpose of working through the various problems on a tablet was not to determine the answer, but to learn various methods of reducing complicated problems to simpler ones. Therefore, the mathematical tablets in general, and the ones on quadratic equations in particular, were used to train the minds of the future leaders of the country. It was not really that important to solve quadratic equations—there were few real situations which required them. What was important was for the students to develop skills in solving problems in general, skills which would give the students the power to solve the problems which a nation's leaders needed to solve. These skills included not only the ability to follow well-established procedures (algorithms) but also the ability to know how and when to modify the methods and how to reduce complicated problems to ones already solved. It was not desirable in Babylonia for all students to learn algebra; there was only a little room at the top of the society. Thus, the study of algebra and, in particular, the study of quadratic equations were used as a filter to determine who would be allowed to become the leaders of the future. An interesting question to answer is what happens when *everyone* truly has the power of thought that the study of algebra conveys.

Let us now move to China around the year 200 B.C. The most famous book of Chinese mathematics written around that time is the *Jiuzhang suanshu* (*Nine Chapters on the Mathematical Art*), a book which soon became a Chinese classic and which prospective civil servants had to master in order to win a secure government job. Like the Babylonian tablets, this book was a compendium of problems, presumably problems of the type government workers needed to solve. It was divided into nine chapters, each dealing with a different type of problem. And like the Babylonian scribes, the Chinese authors included for each problem a detailed solution algorithm but did not discuss how the solution method was derived or why it worked. That was evidently part of an oral tradition, because later commentators often did suggest the possible reasoning of the original authors.

First, let us look at problem 26 of chapter 6, because we see here that "old problems never die": "There is a reservoir with five channels bringing in water. If only the first channel is open, the reservoir can be filled in 1/3 of a day. The second channel by itself will fill the reservoir in 1 day; the third channel in 2 1/2 days, the fourth one in 3 days, and the fifth one in 5 days. If all the channels are open together, how long will it take to fill the reservoir?" (Vogel, p. 68) (This problem, with only the numbers modified, can be found in almost every algebra text to the present day. Look in your favorite one.)

A more interesting chapter for our discussion is chapter 8, which deals with the solution of what we call systems of linear equations. Chinese mathematics was performed on a counting board with small bamboo rods. Thus, a problem involving a system of linear equations would be set up with the coefficients in different squares of the board and would therefore appear as a matrix. The Chinese solution of these problems was by the method now known as Gaussian elimination. Consider problem 1 from chapter 8: "There are three classes of grain, of which three bundles of the first class, two of the second, and one of the third make 39 measures. Two of the first, three of the second, and one of the third make 34 measures. And one of the first, two of the second and three of the third make 26 measures. How many measures of grain are contained in one bundle of each class?" (Vogel, p. 80) In modern terms, the problem can be translated into the system

$$\begin{aligned} 3x + 2y + z &= 39 \\ 2x + 3y + z &= 34 \\ x + 2y + 3z &= 26 \end{aligned}$$

The Chinese procedure, clearly spelled out in the text, is to arrange the coefficients as a matrix:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{array}$$

and then to operate on the columns. The first step is to multiply the middle column by 3 and subtract off twice the right hand column so that the first entry in the middle column becomes 0. One then multiplies the left hand column by 3 and subtracts off the first column so the first entry in the left column becomes 0. The new matrix is then

$$\begin{array}{ccc} 0 & 0 & 3 \end{array}$$

$$\begin{array}{ccc} 4 & 5 & 2 \\ 8 & 1 & 1 \\ 39 & 24 & 39 \end{array}$$

A similar procedure using the second and third columns turns the matrix into

$$\begin{array}{ccc} 0 & 0 & 3 \\ 0 & 5 & 2 \\ 36 & 1 & 1 \\ 99 & 24 & 39 \end{array}$$

from which one reads off the solution for z from $36z = 99$ and then substitutes to determine y and x respectively.

One interesting point here is that one might expect that such matrix manipulations would sometimes lead to negative numbers, even if the ultimate solutions are positive. In fact, problems are presented in this chapter where that happens—and the Chinese give the rules for operation with negative numbers so that the method can proceed. Neither the Egyptians nor the Babylonians dealt with negative quantities at all. One system producing negative numbers, arising out of a problem about yields of various types of grain, is

$$\begin{aligned} 2x + y + z &= 1 \\ 3y + z &= 1 \\ x + 4z &= 1 \end{aligned}$$

for which the matrices are

$$\begin{array}{ccc} 1 & 0 & 2 \\ 0 & 3 & 1 \\ 4 & 1 & 0 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 2 \\ -1 & 3 & 1 \\ 8 & 1 & 0 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 2 \\ 0 & 3 & 1 \\ 25 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

This example perhaps gives a pedagogical hint, coming out of history, a hint which will be repeated in a slightly different context later. The negative numbers were first introduced not to solve equations previously unsolvable, like $x + 3 = 1$, but in the process of solving equations which had perfectly nice positive solutions via an algorithm which was known to work. They were therefore not "made up" numbers, but numbers which had to be used to continue the algorithm. It was thus necessary for the Chinese to figure out how to operate with these numbers. But we should also note that although the problems in this text are stated in real-world terms, they do not appear to be the kinds of problems which would actually arise in a real situation. Although it is not clear if or when Chinese civil servants would need to solve systems of linear equations, they, like their Babylonian predecessors, were required to master the appropriate mathematical techniques.

We now jump forward to ninth century Baghdad. There the caliph al-Ma'mun had established the House of Wisdom to which he called the best and brightest scientists of his domains to begin to establish a scientific tradition in Islam. One of these scientists was Mohammad ibn Musa al-Khwarizmi (780-850). As we noted above, al-Khwarizmi wrote probably the earliest true algebra text, *The Condensed Book on the Calculation of al-Jabr and al-Muqabala*. Note, of course, that our word algebra comes from al-jabr, which the Latin translators left in that form when they translated the book in the 12th century.

Why did al-Khwarizmi write his text? He explains in the introduction:

> That fondness for science, by which God has distinguished the Imam al-Ma'mun, the Commander of the Faithful, has encouraged me to compose a short work on calculating by al-jabr and al-muqabala, confining it to what is easiest and most useful in arithmetic, such as men constantly require in cases of inheritance, legacies, partition, law-suits, and trade, and in all their dealings with one another, or where the measuring of lands, the digging of canals, geometrical computation, and other objects of various sorts and kinds are concerned. (Rosen, p. 3)

Thus, he was interested in writing a practical manual, not a theoretical one. As he noted, "what people generally want in calculating ... is a number,'' the solution to an equation. Thus, the text was to be a manual for solving equations, both linear and quadratic. Let us consider al-Khwarizmi's solution of an equation of the form $x^2 + bx = c$, an equation similar to the Babylonian one considered earlier. Not surprisingly, given that Babylonia and Baghdad are both in the Tigris-Euphrates valley, al-Khwarizmi's solution procedure is virtually the same as that of his ancient predecessors. Like them, he gave the entire solution in words and in the form of an example:

> What must be the square which, when increased by ten of its own roots, amounts to thirty-nine? The solution is this: You halve the number of roots, which in the present instance yields five. This you multiply by itself; the product is twenty-five. Add this to thirty-nine; the sum is sixty-four. Now

> take the root of this which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought. (Rosen, p. 8)

Al-Khwarizmi differs from the Babylonians in that he explicitly justifies the solution via geometry. The justification is essentially the same as the probable Babylonian one. Namely, he begins with a square representing x^2, and adds two rectangles, each of width five (half the roots). The sum of the areas of the square and the two rectangles is then $x^2 + 10x = 39$. One now completes the square with a single square of area 25 to make the total area 64. The solution $x = 3$ is then easy (see figure 2). Today, we would find a second solution for this equation, $x = -13$, but in Islam of this time there were no negative numbers. So for al-Khwarizmi, $x = 3$ is the solution. We should also note that in solving quadratic equations, al-Khwarizmi did not always consider the unknown as the side of an actual square; it could represent any sort of number.

Al-Khwarizmi's book contains much more than the solution of quadratic equations, including material on manipulation with algebraic expressions, some explained by the use of geometry. The text also contains a large collection of problems, many of which involve these manipulations and most of which result in a quadratic equation. But although al-Khwarizmi promised in his preface that he would write about what is useful, very few of his problems leading to quadratic equations deal with any practical ideas. Many of them, in fact, are just about numbers and begin with the statement, "I have divided ten into two parts." There are a few problems concerned with dividing money among a certain number of persons, but these are in no sense practical. For example: "A dirhem and a half is to be divided equally among one person and a certain number of other persons, so that the share of the one person is twice as many dirhems as there are other persons." [Rosen, p. 59] In modern notation, the problem can be written as $1\frac{1}{2}/(1+x) = 2x$. This becomes $x^2 + x = \frac{3}{4}$, whose solution is $x = \frac{1}{2}$, a strange answer to the question of how many "other persons" there are!

Al-Khwarizmi's work does have "applied" problems in it, but most deal with the question of inheritance, the solution of which requires no mathematics more complicated than linear equations but instead requires detailed knowledge of Islamic legacy laws. So why do we need quadratic equations? Al-Khwarizmi seems to have no better answer than the Babylonians. He could not think of any real-life examples which require them. His main goal, then, just like the Babylonian scribes, was to teach his readers methods to solve mathematical problems, methods which could then be applied to other kinds of problems in daily life.

Other Islamic mathematicians continued the work of al-Khwarizmi and developed new algebraic techniques, many of which were not particularly aimed at solving real-life problems but simply at following an idea to its logical conclusion. For example, al-Khwarizmi demonstrated in his text how to multiply together polynomials of the first degree. A mathematician of the 12th century, al-Samaw'al ibn Yahya al-Maghribi, a Jew from Baghdad who converted to Islam when he was about 40, developed the procedure further to encompass polynomials of arbitrary degree. He also developed a technique for dividing polynomials, a technique using a form of a counting board in which the numbers in a given column represented the coefficients of a particular power of the variable. For example, he showed how to divide $20x^2 + 30x$ by $6x^2 + 12$. It is easy to see that this division does not come out even, so al-Samaw'al showed how to continue the process indefinitely. His "answer approximately" is given as

$$3\frac{1}{3}+5\left(\frac{1}{x}\right)-6\frac{2}{3}\left(\frac{1}{x^2}\right)-10\left(\frac{1}{x^3}\right)+13\frac{1}{3}\left(\frac{1}{x^4}\right)+20\left(\frac{1}{x^5}\right)-26\frac{2}{3}\left(\frac{1}{x^6}\right)-40\left(\frac{1}{x^7}\right).$$

Al-Samaw'al then notes that because there is a pattern to the coefficients of the quotient, he can write out any particular term. In fact, he uses the pattern to write out the next 21 terms and ends with 54,613 1/3 $(1/x^{28})$. Al Samaw'al used the notion here of approximating quotients with partial results to develop the first theory of (infinite) decimal fractions. We may draw a pedagogical hint from the history here: It is often a good idea to let students explore the consequences of a particular technique, even far beyond any immediate practical results.

We return to equations, but no more about quadratic equations. In fact, let us move quickly to 16th century Italy and consider cubic equations. Although in the last decades we have learned that Islamic and European mathematicians struggled for over fo centuries to solve such equations algebraically—that is, to find a cubic formula—none of them succeeded. But once we see the solution finally found by Scipio del Ferro around 1510 and first published by Gerolamo Cardano in his 1545 book, *Ars Magna, sive de Regulis Algebraicis* (*The Great Art, or On the Rules of Algebra*), we may wonder why none of them succeeded. As is true for many problems, the solution is easy once it is shown to us.

Let us consider the cubic equation $x^3 = cx + d$. The key to the solution was to consider this as an equation involving solids and to set x as a sum of two quantities, $x = u+v$. Then, since $x^3 = u^3 + 3u^2v + 3uv^2 + v^3 = 3uv(u+v) + (u^3 + v^3)$, it follows that all we need to do is solve the two equations $3uv = c$ and $u^3 + v^3 = d$. These reduce to a quadratic equation in u^3, namely, $u^6 + (c/3)^3 = du^3$, an equation whose solution is easily found. It follows that the solution to the original equation is

$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}} + \sqrt[3]{\frac{d}{2} - \sqrt{\left(\frac{d}{2}\right)^2 - \left(\frac{c}{3}\right)^3}} .$$

We may ask the same question here as we did in the case of quadratic equations. Did cubics arise to solve real-world problems? If we look in ancient sources, where the Greeks solved a few specific cubics, or medieval Islamic sources, where Omar Khayyam used the theory of conic sections to solve arbitrary cubics geometrically, we find that the equations arose out of mathematical questions, like, for example, the classic problems of doubling the cube or trisecting the angle. And even in Cardano's book, most of the problems illustrating the rule are "artificial," even though a few are stated in real world terms, mostly involving compound interest.

The cubic formula is generally not taught in schools today, although it does occur in most algebra texts through at least the end of the 19th century. Its exclusion perhaps should be reconsidered, not because one needs to solve cubic equations, but because, on an elementary level, the formula led to the introduction of complex numbers and, on a more advanced level, it was part of one of the strands leading to the development of group theory. A glance at the formula from a modern perspective shows how complex numbers come in. We note—and so did Cardano—that the formula does not seem to make sense if $(c/3)^3 > (d/2)^2$? For example, what happens if we use this formula to solve the cubic equation $x^3 = 15x + 4$? In this case, although it is clear that $x = 4$ is a solution, the formula gives us

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Cardano was not sure what to do with answers like this, but Rafael Bombelli some 15 years later wrote a new systematic algebra text in which he dealt with this question by introducing complex numbers with their arithmetic defined by normal algebraic rules and the obvious further rule that $\sqrt{-1})(\sqrt{-1}) = -1$. Bombelli could then demonstrate that the two cube roots in the solution above were in fact equal to $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$, so that the formula gave the desired answer $x = 4$. Thus, just as in the case of the negative numbers introduced in China, complex numbers entered mathematics because they were necessary to solve problems with known solutions which could be found through the use of a valid algorithm. They did not arise from trying to solve $x^2 = -1$; that equation simply had no solutions.

Naturally, there is more to the story of complex numbers than this. Mathematicians did not feel comfortable with their use until complex numbers were themselves given a geometric interpretation around the turn of the 19th century. For now, we note that once Lodovico Ferrari, a student of Cardano's, succeeded in solving fourth degree equations, there was no further progress in the old problem of equation solving until the 19th century. There were other important developments in algebra in the interim, of course, including the introduction of modern notation and the study of the elements of the theory of equations by René Descartes in the 17th century. But in our survey of algebra and its texts, we now want to jump to the 18th century and consider the work of Leonhard Euler.

Euler's *Introduction to Algebra* was the most complete work of its kind, summarizing all of the work accomplished in the solution of equations over the centuries. The text dealt first with the properties of integers, fractions, irrational numbers, and complex numbers; it then continued with manipulation of algebraic expressions, including material on converting fractions to infinite series and on the extraction of roots of polynomials; Euler then went on to discuss ratio and proportion and concluded with a long chapter on the solution of polynomial equations. All of this material, however, dealt with numbers of one sort or another. The manipulation of symbols all came out of the standard rules for manipulating numbers, and the symbols themselves always stood for numbers.

The central point to make about Euler's text, in contrast with those from previous centuries, is the total lack of diagrams. Euler was the master analyst, and in his text on Differential Calculus of 1755 he insisted that because it was a work on analysis it would contain no diagrams whatever. Analysis dealt with the manipulation of numerical quantities, and although some ideas may have come originally from geometry, it is not necessary to use geometry in a rigorous work on the subject. Euler considered his work on Algebra a prerequisite to his calculus material, and so he continued this restriction. Thus, for example, although he dealt with the method of completing the square in order to solve quadratic equations, Euler's derivation of this method was entirely algebraic.

It is also interesting to see Euler's understanding—or lack thereof—of complex numbers at this time, before there was a geometric representation. Euler introduces imaginary numbers (e.g., $\sqrt{-4}$) as numbers which exist in our imagination, but which have the property that, multiplied by themselves, they give negative values: $(\sqrt{-4})(\sqrt{-4}) = -4$. "Moreover," Euler continues, "as $\sqrt{a}$ multiplied by $\sqrt{b}$ makes $\sqrt{ab}$, we shall have $\sqrt{6}$ for the value of $\sqrt{-2}$ multiplied by $\sqrt{-3}$ and $\sqrt{4}$, or 2, for the value of the product of $\sqrt{-1}$ by $\sqrt{-4}$!" (Euler, p. 43)

Mistakes notwithstanding, Euler's book provides a wide variety of problems to illustrate various algebraic topics. In particular, he makes a special effort to find real-life problems which result in linear and quadratic equations, many of which date from long before his time and continue to appear in newer books. Let us consider a few:

- One rents 25 acres of land at 7 pounds 12 shillings per annum; this land consists of two sorts and one rents the better sort at 8 shillings per acre, the worse at 5. Required the number of acres of the better sort. (Euler, p. 204)
- The price of 1 acre of good land is 300 pieces of gold; the price of 7 acres of bad land is 500. One has purchased altogether 100 acres; the price was 10000. How much good land was bought and how much bad? (From *Jiuzhang suanshu*) (Vogel, p. 78)
- A person rents 25 acres of land for £7 12s per annum; for the better part he receives 8s. an acre, and for the poorer part, 5s. an acre; required the number of acres of each sort. (Brooks, p. 134)
- Three brothers bought a vineyard for a hundred guineas. The youngest says, that he could pay for it alone, if the second gave him half the money which

he had; the second says that if the eldest would given him only the third of his money, he could pay for the vineyard singly; lastly, the eldest asks only a fourth part of the money of the youngest, to pay for the vineyard himself. How much money had each? (Euler, p. 213)

- Four men have money and intend to buy a horse. The first could buy it with his own money plus 1/2 of that of the second, 1/3 of that of the third, and 1/4 of that of the fourth. The second could buy it with his own money plus 1/4 of that of the third, 1/5 of that of the fourth, and 1/6 of that of the first. The third person requires his own money plus 1/6 of the fourth, 1/7 of the first, and 1/8 of the second. Finally, the fourth requires his own money plus 1/8 of the first, 1/9 of the second, and 1/10 of the third. How much does each have and how much does the horse cost? (From the *Liber abbaci* (1202) of Leonardo of Pisa) (Boncompagni, p. 349)
- A horse-dealer bought a horse for a certain number of crowns, and sold it again for 119 crowns, by which means his profit was as much per cent as the horse cost him; what was his purchase price? (Euler, p. 226)
- Mr. Leslie sold his horse for $171, and gained as much per cent as the horse cost him; what was the first cost of the horse? (Brooks, p. 219)
- A jockey sold a horse for $144, and gained as much per cent as the horse cost. What did the horse cost? (Wentworth, p. 218)

Because not much has changed in what we can call elementary algebra since the time of Euler, the proofs of the first three proposed theorems are complete. They were intended to apply to elementary algebra, not to modern abstract algebra. For the fourth one, however, we need to continue our survey and consider the great changes which took place in algebra beginning in the 19th century, changes which led to our modern abstract algebra. Although elementary algebra texts in the 19th century continued to deal with much of the same material as had Euler, the continued search for solutions to polynomial equations was one of the central factors which led mathematicians to greater abstraction, in particular, to the concept of an abstract group.

Many mathematicians after the 16th century attempted to generalize the methods of Cardano and Ferrari to solve algebraically polynomial equations of degree higher than four. In the late 18th century, Louis Lagrange began a new phase in this work by undertaking a detailed review of these earlier solutions to determine why the method for cubics and quartics worked. As part of this review, he became interested in the study of permutations of the roots of an equation. Although Lagrange ultimately neither found a formula to solve the fifth degree equation nor proved that no such formula existed, Niels Henrik Abel accomplished the second option in the 1820s, using Lagrange's ideas on permutations. Evariste Galois shortly thereafter again made use of the set of permutations of the roots of a polynomial to determine the conditions under which a polynomial equation could be solved algebraically.

It was Arthur Cayley who first noted in 1856 that one could generalize Galois' notion to any set of operations on a (finite) set of quantities. Cayley thus defined a group to be

> a set of symbols, 1, α, β, ... all of them different, and such that the product of any two of them (no matter in what order), or the product of any one of them into itself, belongs to the set...It follows that if the entire group is multiplied by any one of the symbols, either on the right or the left, the effect is simply to reproduce the group. (Cayley, p. 41)

Cayley assumed that the number of symbols was finite and that multiplication was associative. He then introduced a group table to illustrate the multiplication rules.

The theory of groups had other roots as well, in particular in number theory. Thus, Carl Gauss had studied both quadratic forms and residue classes and had noted that many of the properties of both under composition were identical. Other mathematicians noted similar properties in other sets under composition. But it was only in 1870 that Leopold Kronecker finally saw that an abstract theory could be developed out of the analogies.

> The very simple principles on which Gauss' method [of composing quadratic forms] are applied not only in the given context but also frequently elsewhere, in particular in the elementary parts of number theory. This circumstance shows ... that these principles belong to a more general, abstract realm of ideas. It is therefore appropriate to free their development from all unimportant restrictions, so that one can spare oneself from the necessity of repeating the same argument in different cases....The presentation gains in simplicity, if it is given in the most general admissible manner, since the most important features stand out with clarity. (Wussing, p. 64)

Kronecker thus began to develop the simple principles of the theory of abelian groups and went on to prove what we now call the fundamental theorem of Abelian groups.

It only remained for someone to unify the notions of group appearing in these areas and in geometry as well. Not surprisingly, two different mathematicians accomplished this in 1882—Walter von Dyck and Heinrich Weber. And in 1895, Weber published the material on groups in his 1895 text, *Lehrbuch der Algebra*. It is with this publication that the abstract concept of a group can be considered to have become part of the mathematical mainstream. Weber's text also included the definition of a field, again based on many examples that had been considered over the 19th century. Virtually all of the examples of groups and fields, however, were still related one way or another to the real and complex numbers.

Abstraction increased dramatically in the 20th century as mathematicians began to see examples of groups and fields, as well as rings and algebras, in more and more places in mathematics, well beyond the realm of "numbers". But it was Emmy Noether who taught the mathematicians in Göttingen in the 1920s that algebra was central to mathematics, that its ideas extended to all areas of the subject, and that an abstract approach was the way to look at algebraic concepts. In fact, Pavel Alexandrov noted in 1935 that it was Emmy Noether who

> taught us to think in terms of simple and general algebraic concepts—homomorphic mappings, groups and rings with operators, ideals—and not in cumbersome algebraic computations; and she thereby opened up the path to finding algebraic principles in places where such principles had been obscured by some complicated special situation. (Dick, p. 158)

Saunders MacLane has even asserted that "abstract algebra, as a conscious discipline, starts with Emmy Noether's 1921 paper, 'Ideal Theory in Rings.'" (MacLane, p. 10)

The first "modern" text in algebra, van der Waerden's *Modern Algebra*, which appeared in 1931, was heavily influenced by Emmy Noether. It is an enlightening exercise to compare this work with algebra books of just a few decades earlier to see the profound influence that she had on our present conception of algebra. Nevertheless, even Noether realized that one needs to be familiar with a wide variety of concrete examples from all parts of mathematics before one can understand the value of the generalizations she was able to make. Our fourth theorem is thus proved. Its immediate corollary is that one cannot understand the modern algebra of the 19th and 20th centuries without mastering the algebra of previous centuries and, indeed, without understanding material from other branches of mathematics as well.

There is much more one could say about the history of algebra and its teaching in the 20th century. One finds, however, that a typical 20th century elementary algebra text is not far different from the text of Euler, except that it usually covers some analytic geometry and leaves out much of his more advanced material on equations. Naturally, 20th century university texts in abstract algebra are far different, because they all owe much to van der Waerden and Emmy Noether.

I hope that the historical theorems set out at the beginning are useful in the task of reforming the teaching of algebra and that the lessons of history will serve as a guide in this important undertaking.

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Long-Term Algebra Reform: Democratizing Access to Big Ideas

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> *The way curricula are currently specified, 'teaching algebra' is like teaching people to speak by making them move their mouths in certain positions over and over. Ridiculous! We don't teach people to walk by moving their legs for them, or forcing them to put their feet into certain positions: walking is an expression of the urge to be elsewhere.* —John Mason

Overarching Curricular Goals: Democratize Access to Big Ideas and Prepare for Continuing Change

Discussion of the reform of algebra can, and all too frequently does, reduce to quibbling over details ("Is rationalizing of denominators still important?") that probably don't matter given the scale of reform needed to deal with the large scale failure that school mathematics is. Some of the old arguments against change, especially at the detailed level, get harder to pose when we change the terms of the debate—and it is time to do exactly that. In order to deal with the structural factors that limit student access to the big ideas of mathematics, we must attend to the global structure and content of the larger curriculum in which algebra plays an important part. We cannot discuss algebra in isolation from its curricular context.

Replace the Layer-Cake by a Strands Organization of the Curriculum: Respect the Intrinsic Embeddedness of Algebraic Activity

The *Standards* leave the current global structure of the mathematics curriculum largely untouched, especially at the secondary level—the arithmetic, algebra, geometry, precalculus, calculus layer-cake, while enriched, is intact. "This layer-cake approach to mathematics education effectively prevents informal development of intuition along the multiple roots of mathematics. Moreover, it reinforces the tendency to design each course primarily to meet the prerequisites of the next course, making the study of mathematics largely an exercise in delayed gratification. To help students to see clearly into their mathematical futures, we need to construct curricula with greater vertical continuity" (Steen 1990, p. 4).

In addition to the layer-cake problem, we need to confront the basic fact that the secondary school mathematics needed in the 21st century will be drastically different from today's, encompassing many more complex and abstract ideas than we now consider critical. It is useful to recall how the mathematics of the 18th and 19th centuries is subsumed by the mathematics of today as special cases of more general and abstract ideas. The process of growth has not stopped, it is accelerating. As a consequence, the elementary and middle school curricula will need to do far more for children than our one curriculum does now, which is to prepare the college-intending for the top of the layer-cake, formal calculus, and the mathematics of classical physicists. Other students, who cannot survive the rigors of an abrupt introduction to character-string manipulation at puberty, have largely been ignored, shunted into curricular and occupational dead-ends.

With Steen and others, I suggest a strands organization, where major ideas weave through many grade levels, frequently interweaving with one another to create a rich fabric, but one that has a direction, a natural flow, from the wide watershed of concrete experience to generalizations and abstractions, from informal and language-based representations to more formal representations. Filters are usually built using layers, while strands provide a natural flow, gradually drawing into mathematics ever more diverse experience. Aspects of this approach were suggested by Jerome Bruner about three decades ago (Bruner 1960), but the new ingredient, beyond stark necessity, is increased emphasis on the need for vertical coherence and integration among strands. We have seen how easily Bruner's spiral sags into a circle if we do not attend to building authentic progression of mathematical content.

So what are candidates for strands? Some very thoughtful mathematicians have already offered suggestions in the book cited above: quantity, shape, uncertainty, dimension, and change. Interestingly, algebra does not appear in the list, but, rather, is implicitly included as an expressive language in the development of the strands listed. Whatever the strands, they should begin in elementary school and contribute to a core mathematical experience for all students up through the first years of secondary school, with variations expected and encouraged. For the latter years of secondary school, we should expect real differentiation and choice. The big ideas will lend a coherence and purpose to K-8 mathematics that is not currently present, even in reform documents. Although, to be fair, recent NCTM *Standards* addenda are beginning to introduce some major ideas at the lower levels, although without much technological support and in a piecemeal way.

Planned variation at grades 11 and 12 will serve several functions beyond offering choice to different students with different wants and needs. The high school curriculum can become a place for genuine intellectual expression on the part of teachers, with opportunity and reward for innovation. High school mathematics at this point suffers the profound alienation of an aged assembly line that has outlived any productivity it ever had. Teachers burn out in huge numbers after only a few years of teaching the unwilling the unwanted. The carrot is wilted and the stick doesn't work. But curricular diversity not only plays to a traditional American strength, enlarging the space in which inventiveness and choice can thrive, but it can also be made to fit the coming telecommunications revolution. Politicians and policy people speak of choice, but there is not much curricular choice: It's choice between factoring x^2-4 and factoring x^2-9. We need a new, more diverse curriculum deliberately designed to grow and change.

Eliminate Algebra as a Set of High School or Early College Courses and Integrate the Basic Ideas of Algebra into K-8 Mathematics—"De-coursify Algebra"

The mathematics of all content strands should grow out of acts of sense-making. Students develop mathematical ideas in order to render one's experience knowable and communicable. They build and refine models that capture experienced patterns and lead to new patterns at higher levels of abstraction. But they begin by tapping fundamental intuitions and building on natural ways of apprehending, organizing, navigating and communicating. The models are developed using ever more powerful and systematic languages and representations, beginning with physical materials and iconic and graphical forms but leading to an array of efficient languages that include potent syntactical structure embodied within computational media. This means, among other things, algebra should be learned as a sense-making tool throughout elementary and middle school. The two sides of algebra discussed below (algebra as an expressive language versus algebra as a set of structures) can both be addressed in K-8. Indeed, grade 5 materials from some new curriculum development projects include structural algebra (Romberg, in press).

Relative to algebra as a language, we can informally distinguish two levels of syntax. One is associated with writing and parsing expressions (mainly functions) or simple equations. The other is associated with the transformation of expressions or equations into different forms (often, but not always equivalent forms). The former level should be the province of grades K-6, the latter for grade 7 and beyond (but see the suggestions regarding learning-as-needed skill modules below). Much curricular material has been produced over the years in the United States and elsewhere, although it has not been integrated with a full strands-oriented, big-ideas curriculum.

We need to develop algebra as a tool for representing rates and differences of quantities, and the relations with accumulations of those varying quantities. We have tended to treat early algebra apart from the mathematics of change, more in the spirit of the uses of algebra that preceded the development of calculus than in the spirit of the uses that developed later. The rhetorical roots of algebra, with their classical problem types (Katz, this volume) may not be the best context for the curriculum of the next century. We may need to concentrate on more recent applications of algebra to build toward the big ideas that should be part of the mathematical experience of all students. An important factor in this reconsideration of the place of algebra relative to the mathematics of change is the ability of technology to provide numerical and graphical solutions to difference and differential equations as well as the reverse—to differentiate "totals" data, providing rate information in graphical and numeric form.

Eliminate Calculus as a Capstone Course for the Elite by Building a Strong Change-strand into K-10 Mathematics: Calculus for All

If, politically, algebra for all is attractive, why not calculus for all? And then, why not serious mathematics for all? More importantly, the mathematics of change can be a prime context for developing algebra, just as it was historically (Kaput 1994). We should caution, however, that by calculus I do not mean the kind of formal technique-oriented exercise represented by traditional university calculus, but a broader and deeper study of the relations between varying quantities and the accumulation of those quantities, beginning with concretely based numerical and graphical problems and progressing to more algebraically represented or representable problems. (See the next section.)

Bring New Topics Into the Core Grades 9-10 Curriculum, Including Linear Algebra, and Especially a Wider (Iterative) Mathematics of Change, of Uncertainty, of Space, of Proof

Reconsidering the place of algebra in a revised core curriculum requires us to acknowledge (1) the changing nature of school algebra and (2) the changing curricular context for that algebra—the other topics that may deserve a place in the grades immediately following middle school. While the languages of algebra include the basic character-string system, coordinate graphs and tables, they should not be limited to the traditional relations over sets of real numbers and accompanying coordinate graphs. They should include matrices and variables over non-numerical objects and operations on them of the sort traditionally associated with abstract algebra (strongly concrete to begin with, of course, as in the Connected Mathematics Project middle school materials originating in the Dutch curriculum; Romberg, in press). In addition, the languages of algebra, already established in their basic representational role in grades K-8, should be applied to the different and evolving ways of representing statistical information as well as data with other structured forms. Indeed, the basic idea of data structure is likely to be important in the next century. Multidimensional and dynamic graphics will likewise grow in importance. Already, we have seen how spreadsheet representations have heightened the importance of iterative and recursive forms for representing functions.

We can expect similar evolution in the mathematics of space, leading to geometry on hyperboloids or ellipsoids, and Euclidian geometry of many dimensions, and fractional dimensions. It is easy to imagine students building a construction and then varying the curvature of the space in which it appears. Or examining finite geometries. Or studying transformational geometries from an algebraic perspective using appropriate computer software. Students are not learning 2-D Euclidian geometry today. Indeed, they are more often than not being given a lifelong inoculation against geometry. They should be given the opportunity to learn basic ideas of analysis in the context of the mysteries of the infinite—especially if these mysteries had been explored on occasion in earlier grades, when, as any parent will attest, kids approach them with almost infinite intensity and wonder.

Students need to be put in the position where conjecture, argumentation, and proof occur as authentically experienced means for determining what is true. It is evident among today's students at all levels, including mathematics majors at elite institutions (Artin, this volume), that the idea and purposes of proof are ill-understood.

Finally, we should continue to evaluate the central concepts that will help organize the important ideas of school mathematics—for example, the ideas of sensitivity, orbit, closeness, order, optimization, randomness, quotient and product structures, sameness of structures, generators and independence, algorithm, robustness, computability, model. Further, we need to organize these central ideas. Some are algebraic in character, some topological, some computational, some statistical, while others, such as orbit and model, cut across or transcend mathematical topics. The importance of selecting and organizing key organizing principles grows as students' mathematical experience diversifies.

Create a Mathematics Elective Structure for Grades 11-12 to Confront Both the Diversity of Students and the Diversity of Useful and Important Mathematics

The answer to student diversity is curricular and pedagogical diversity. There is no shortage of mathematics for students to explore, and clearly, no student will be able to get a full sample in the core curriculum through grade 10. One answer to both sides of the diversity challenge is an elective structure for mathematics in grades 11 and 12, coupled, as argued below, with an incentive structure to promote innovation and dissemination. Rather than present a wide-open menu, we will need to organize the offerings (a) by student objectives (e.g., college-intending versus vocational) and (b) by key mathematical ideas. Within the college intending and vocational categories, further subdivision is likely to be appropriate—for example, for college-intending, technical/business/nontechnical might suffice. Obviously, the college admissions process will need to be adjusted, although not drastically. The most important and immediate adjustment needed is in the content of what is tested, for both college admissions and introduction to the world of work—away from surface computational skill and toward reasoning and problem solving. Later, the content of introductory college mathematics courses will need to be revised. However we choose to change, to stay in the rut we inherited is simply indefensible.

Create a Broadly Available Modular Skills-on-Demand System

Currently, our algebra courses have manipulative skills at the center and reasoning and applications at the edges. Most of the skills have special uses among particular subgroups of the population later in their education or careers. We need to break the paralysis that attends the entirely legitimate worry that students will be seriously harmed by lack of particular skills. One way to do this is to make those skills available, using Intelligent Computer Aided Instruction, on a universal basis. Learning is always more effective when its purposes and uses are experienced as immediate. The technology of adaptive testing and ICAI is ideal for delivering particular skills on an inexpensive flexible basis—in connection with college courses, in the service of vocational training in the private or public sectors, or in connection with other K-12 courses (including nonmath courses). Well-designed modules might serve different purposes depending on who uses them.

Learning Algebra as an Expressive Language: Algebraic Objects That *Do* Something

The computational medium allows us to reverse the inferential arrow from experience to algebra. We now can use algebra to control a physical phenomenon (or a simulation of such), thereby generating a regularity in physical experience, a phenomenon in some reference field. This appears to be a new use of the medium, although it bears close relation to the idea of programming. It is not yet obvious how to exploit this opportunity, but it seems that one can apply one's knowledge of the semantics of the reference field to create, manipulate or make judgments about algebraic objects. For example, if we are modeling the position of swimmers in a multi-lane swimming pool, what does it mean algebraically to have a temporal versus a spatial headstart, or to travel in opposite directions? Thus, for example, a student might determine the distance between two swimmers at a certain point, determine the difference in their speeds, and then use that "closing speed" to segment the distance to find the catchup time. The key is that performing actions on algebraic expressions, as well as setting up equations and inequalities, take on meaning in the reference field(s) for algebra. They also afford rich opportunity for conjecture and experimentation that exploit the student's knowledge of the reference field to make judgments about his or her algebraic actions.

Using algebra to control phenomena involves a different side of language, different from the descriptive or representational side and more on the expressive side. It is the side that embeds language in action, in activity. Indeed, the phenomena that result from executing or running algebraic objects can be thought of representing the algebra rather than the other way around.

Some Enabling Goals and Actions

Most of the following goals relate to one or more of the following three assumptions: (a) big ideas take a long time to develop and interconnect so should be approached as curricular strands, (b) we need to take technology much more seriously than we have previously, and (c) diversity and richness in curriculum is to be valued over uniformity and linear sequencing. Much of what is discussed here is not strictly confined to algebra, but bears directly on algebra reform nonetheless.

Change the Prevailing Conceptions of Algebra, Drop the Idea of Algebra as a Course, and Begin Thinking of Symbolic Algebra as a Particular Form of Representation Among Others

We need a better view of school algebra out there among the lay population (this includes politicians, teachers, school administrators, even some mathematicians). Perhaps we should begin thinking of traditional symbolic algebra as but one among many means for expressing quantitative relations, each with its own strengths and weaknesses (Confrey 1992; Dennis & Confrey, in preparation). Indeed, while strides are being made based on new developments to increase the public's awareness of mathematics as a live and important part of our culture, it is at least as important to widen our collective conception of traditional subjects. Subjects such as algebra and calculus are identified with the narrow technique-oriented course experiences by those who have taken them, and as some kind of unattainable priestly knowledge by those who have not. Of course, neither identification is close to the truth. Long-term success of deep reform of the sort being advocated in this paper requires this cultural transformation.

Move From Layer-Cake Reform to Strands-Systemic Reform

Historically, we have tended to approach the reform of curriculum under the influence of the inherited layer-cake mentality and from the perspective of the layered communities that constitute the educational establishment. Layer-cake reform implies more layer-cake curriculum. We need to take a much more vertically holistic view of mathematics learning, and a view that suggests building "vertical teams" of innovators rather than grade level-based teams. While we have intended to think "systemic" horizontally, we should probably begin to think in terms of the vertical dimension as well (as has the UCSMP). We may also want to think about developing multiple strands—the more connected the fabric, the stronger it is. More generally, long-term, large-scale curriculum development is in order. In addition, we need new mechanisms for curriculum development (see the discussion of incentive structures below).

The Case of Calculus Reform. The calculus reform initiative offers perhaps the clearest example of how to avoid serious change of the larger K-16 system. While leaving as much educational structure untouched as possible, work on the top layer, but, in the words of every rock-climbing instructor, "whatever you do, don't look down!" I should note, however, that as long as we can avoid saying that reform of calculus is now "complete," we may be able to use the considerable good work that has been done to revise how the formal side of calculus is handled across a wider range of age levels.

The Case of Algebra. The background assumption of most of the algebra reform talk, including "algebra-for-all" talk, remains that algebra is a series of high school courses that either we must "fix" or do a better job of preparing for. I believe we need to go beyond pre-algebra in the 7th grade followed by some kind of algebra experience in 8th grade. As with calculus, much of the recent good curriculum development work could serve to help integrate algebraic reasoning into the earlier grades. And some of that work (e.g., in Romberg's Wisconsin/Utrecht middle school project) is already including some algebra in earlier grades.

The Case of Arithmetic and Rational Number. We have had a long history of innovating in elementary arithmetic independently of our innovating in higher mathematics. But, in a strands approach, the mathematics of earlier and later grades are not independent. The "change" problems of K-2 mathematics need to be seen as part of the change strand and generalized. This is the approach taken in the new TERC K-5 "Investigations" materials (Russell, in press). The same could be said of the extensive work on rational number and proportional reasoning which until recently—with the exception of some partial attempts by Karplus in the early seventies—ignored the function idea and the notion of linearity.

Get Serious About Technology!

William Blake spoke of the most powerful constraints of all, those of the imagination—"mind-forg'd manacles." Most technology uses that have been part of the reform discussion have centered on improving what we are now doing rather than changing in fundamental ways what we are trying to do. Following are a few illustrations of technology applications that have not appeared in NCTM *Standards* discussions to date, and that are less defined by new mathematical content than by the new potentials of the technologies themselves. We will concentrate on uses of technology that have not been widely discussed, so we will not pursue some now-familiar applications such as hot-linkable representations, off-loading of routine numerical or symbolic computations, and enhanced actions on graphical objects.

Acknowledge the Fact That the Medium for the Mathematics of the Next Century Will Be Computational. We are at the threshold of a transformation of the nature of mathematics. Soon, the question regarding whether or not to use computational media will be moot. You can't study dynamical systems, create simulations, or manage large amounts of data in inert media. How many people walked to the Algebra Initiative Colloquium conference? We need to reorganize our expectations and ways of thinking about the content of mathematics to make room for the dramatic changes and expansion in mathematics that will be part of tomorrow's world, our students' world. The two orders of magnitude increase in transportation speed have been incorporated into our travel expectations, but such technological improvements have not been incorporated into our educational expectations.

Get Systematic in the Pursuit of New, More Learnable Representations of Important Mathematics—Technology Has One of Its More Important Applications in This Area. The fact that average youngsters can do multiplications that once required Roman specialists using calculi cannot be attributable to improvements in nutrition or improvements in the gene pool. It is directly attributable to the improvement in the notation system that functions as a cognitive amplifier, a cognitive technology—just as other technologies amplify or modify other human powers. We should examine ways to do the same thing for calculus. Graphically oriented approaches might serve to democratize access to such big ideas as the mathematics of change, especially if they can be hot-linked to student controllable phenomena—either simulations or real-time data collection as with MBL devices (Nemirovsky 1993; Thornton 1993; Kaput, in press). Given that much algebra instruction has been aimed at preparing for calculus, the implications of such changes for algebra are considerable.

Concretization of the Abstract. In mathematics education we have only begun to exploit the varieties of new representations that can assist visualization, but scientists in virtually every field are rapidly developing new ways of seeing. Again, the underlying strategy is to employ the human brain's well-developed natural powers.

Changing Relations Between the Particular and the General. Jere Confrey's Function Probe software (Confrey 1993) includes a screen calculator that keeps a record of keystrokes that one can edit by substituting a literal for a number, so that a particular numerical computation becomes a function of one variable—which can then be used to fill in a table, be graphed, and so on. Several geometry software environments store one's constructions as replayable, and even editable, procedures. One acts on particulars, but a generic object results. It is now commonplace to vary the shape of a function's graph and see its parametric representation change as one performs the transformation. These are simple examples of the new relations between the particular and the general that computational media can provide.

Change the First Two Years of Collegiate Mathematics to Match the New K-12 Curriculum—the Strands Should Continue

The colossal waste of "remedial mathematics" described earlier should not be allowed to continue, especially at the 2-year colleges. The curriculum of the first 2 years of college mathematics should embody the same principles that the new grade 10-12 curriculum does. And it may need to—the consumer-oriented attitude of admissions offices has not yet reached into mathematics—our current curriculum of the first 2 years embodies the same choice that is available in the high school math curriculum—about the same as Henry Ford offered of the model A: "You can have any color you want, as long as it's black."

Stepping Back for a Wider Perspective

Rethinking the place of algebra in school mathematics requires us to take a longer view of the evolution of algebra and its relations to the evolving media in which it has been and can be expressed. In order to do this, we need to get some sense of the different sides of algebra as well as the different characteristics of the media in which we do algebra. There is a tendency over time toward representational pluralism in mathematics—historically, we find ever more diverse ways of representing its concepts. Computers, especially with their powerful graphics systems, aid and abet this tendency. Historically, algebra was a character-based symbolic system designed for the efficient expression of, and action on, quantitative relationships and patterns, but is now a web of representation systems that are physically linkable in the computer medium. Hence the language side of algebra is expanding, mainly in graphical-visual ways, but more on this later.

Form/Content Duality in Mathematics and Algebra

Algebraic reasoning has a dual nature, reflecting a duality of mathematical thinking more generally. Mathematics is both a web of languages, for the expression of pattern, structure and argument (Steen 1988) as well as the content, or referential object, of that pattern, structure and argument. The relation between these two aspects is complex, recursive, and varies according to the nature and purposes of the mathematical activity at hand. That is, it is not fixed. We can focus explicitly on the structure and treat it as the referential object of expression (as when we construct, say, an abstract quotient object of some kind). On the other hand, the patterns expressed by mathematics are typically used as the means to express other mathematics (Kaput 1987) (as when we say Z is a principal ideal domain, or when we define a functor from one category to another), or to express other patterns in experience (as when we use the factor-structure of Z to decide which sets of marbles might be arrayed as rectangles). Moreover, we may switch between usages with the same fluency that we switch between looking through a window and at a window—often with the same lack of outward evidence. The difference is a subtle, but real change within the actor. Sometimes we use the symbols or representations as things-in-themselves, with their own formal rules and structures, and sometimes we see through them to other patterns in our experience. And if this complexity weren't enough, mathematics constitutes many different languages and, of course, many different patterns and layers of patterns. Real mathematical thinking usually involves rapid navigation among all these.

Reconsidering Relations Among Arithmetic, Quantitative Reasoning, and Algebra

We have had a long history, dating back to the 18th and 19th centuries, of regarding algebra as generalizing the properties of arithmetic. And it certainly does. Importantly, this historical and logical/formal relationship between arithmetic and algebra has also been the basis for the curricular relationship. However, this relationship deserves another look, particularly in view of what we are coming to learn about the cognitive foundations of each in quantitative reasoning (Greeno 1989; Greer, in press; Resnick 1992; Thompson 1993, 1994). In order to make sense of the analysis, we need to distinguish between arithmetic and quantitative reasoning.

I define arithmetic to be computation with numerals, usually organized by the structure of the (now) standard numeration system. Much of our elementary mathematics curriculum is centered on teaching arithmetic in the sense of this definition. Abstract quantitative reasoning involves what we usually take the numerals to stand for, as when a child determines how much more 33 is than 18 by counting up from 18, perhaps using fingers to help monitor the results. Another quantitative reasoning approach might involve separating the difference into the 10-part between 20 and 30, and the two smaller parts of 2 and 3 on either end of the difference, and combining these mentally. An arithmetic approach would be to write the 33 - 18 subtraction statement, probably in vertical form, and then use a standard subtraction algorithm to get the result. The key is that the arithmetic approach is mediated by one's knowledge of the rules for operating on the symbols. While a shorthand way of phrasing the distinction might be that quantitative reasoning is "mental arithmetic," I would want to exclude that type of mental arithmetic that involves mentally manipulating images of the numerals.

For an abbreviated characterization of situation-based quantitative reasoning (a full exposition is well beyond the scope of this paper), I paraphrase P. Thompson (1993, 1994). (Situation-based) quantitative reasoning is the analysis of a situation into a network of quantities and quantitative relationships. Quantities are conceptual entities that are constituted in people's conceptions of situations. A person is thinking of a quantity when he or she conceives a quality of an object or event in such a way that this quality is measurable or countable. A quantity ... is composed of an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality. For example, we might think about heights of people we know, speeds of cars we see, ride in or drive, or masses and densities of blocks of wood. While we can in principle assign a numerical value to a quantity, we needn't do so in order to engage in quantitative reasoning. For example, I might make ordering-judgments about the differences between my height and that of my brother and sister without actually performing measurements or looking them up in medical records. This would constitute primitive situation-based quantitative reasoning.

I offer the following assertion, put in simplest terms: Both arithmetic and algebra provide formal means for the externalization of quantitative reasoning. In the case of arithmetic, these means serve the purpose of computing or deriving specific values of quantities. In the case of algebra, these means serve the twin purposes of (1) generalizing and abstracting quantitative relationships and (2) reasoning with the external representations of those quantitative relationships. In some cases, the reasoning is directed toward finding particular values (as in equation solving), while in other cases it is directed toward making explicit certain relationships that were previously implicit by changing the form of algebraic expressions or equations/inequalities. The changed forms better support comparisons or efficient computation.

While our current curriculum assumes arithmetic must precede algebra, so that it can serve as the basis for algebra's more general rules, we should consider the possibility of building algebraic reasoning more directly out of quantitative reasoning. Currently, we use quantitative reasoning mainly as an excuse to employ arithmetic—we create situations where the student must build quantitative relationships so that the given numerical information can be input into these relationships and a computation performed leading to a numerical answer (although sometimes the situation-based reasoning leads to abstract quantitative reasoning—mental arithmetic—rather than arithmetic). However, these problems have been peripheral to the main purpose of the early mathematics curriculum, which is to develop knowledge of arithmetic. Later, much later, we introduce algebra as generalized arithmetic in order to legitimatize algebra's syntactical rules, train students in the use of these rules, and then return to the use of algebra to model quantitative situations, usually in order to determine some value that "solves" the problem (algebra in its ancient sense)—rather than to provide a general model to reason with.

The early history of algebra, predating the development of sophisticated arithmetic systems is revealing. Essentially all the problems that are associated with the early rhetorical forms of algebra were basically quantitative reasoning problems. However, it was not until Vieta that the external symbolism reached a point of supporting reasoning (Klein 1968). Until then, various techniques were used to off-load the complexity onto some extracortical devices, including geometrical figures, and the generality of either the problems attacked or the techniques used was quite limited. Algebra is much more today, not merely due to the development of reliable syntax and number systems, but also because of the development of Western science, the goal of formalizing patterns in data, and the emergence of algebra as the language of science (Bochner 1966).

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Appendix

Ten Reasons Why Early Algebra Makes Sense

1. Tracking: Algebra is currently the primary vehicle for the tracking system at the upper middle and high school levels. The closer algebra is to graduation, the more effectively it can act in its negative filtering role.

2. Algebra as a *course* is simply wrong: As an *expressive set of languages* it needs to be used for many years across many situations, as a means to extend and generalize quantitative reasoning and to serve the development of multiple strands of mathematics. As an *important set of structures*, it should grow out of and help contextualize arithmetic, and later (in high school) should grow out of work with polynomials, matrices and other objects in a spirit of conjecture and justification, setting the stage for further learning at the college level. The intrinsic embeddedness of algebraic reasoning is violated by treating it as a course.

3. Technical education: The same general quantitative sense-making abilities fostered by a strands approach, with elementary algebra integrated throughout in K-8 (e.g., in change, statistics, space), would serve both the needs of a technical work force and the college-intending. The differences that would emerge at the secondary level would be largely in the degree of formalization of the concepts and intuitions already established. The college-intending would have these concepts increasingly formalized and extended in the language of mathematical formalisms, hopefully connected with important scientific ideas. In technical education, we would be freed of the need to treat algebra and other mathematics in formal ways and instead concentrate on embedding those ideas in different concrete situations.

4. To prepare students for the 21st century we absolutely need the curricular space that is now locked up by the 19th century high school national curriculum.

5. We absolutely need space for a green and growing edge at the high school level: We need to build continuing renewal and diversity into the upper levels of the curriculum.

6. Reform is cumulative: To do what needs to be done at the high school level, we must begin serious reform early (as with the Cognitively Guided Instruction Project) and keep building in a longitudinally coherent, strands-oriented manner.

7. Early algebra has worked in other countries (e.g., Russia, Eastern Europe, and the Netherlands).

8. Generalizing and expressing generality is an inherent part of human knowing/learning from the beginning, and language learning is easier for younger kids than older kids.

9. New uses of technology, graphical and symbolic, make many new representations and approaches possible—and necessary. These new approaches offer the opportunity to rethink the place of routine skill development as well as the degrees of symbolic formalization of major mathematical ideas. New principles become available for organizing the curriculum.

10. We need to balance the need for deep long-term change with shorter term risks and constraints. A major risk of tinkering at the margins is to squander the spirit of reform and to ratify for the next generation a curriculum that is only a minor improvement over what we have today. We have mounting evidence that young children can learn much more mathematics than they now are asked to, and we know they will need to know much more in the next century.

Algebra in the K-12 Curriculum

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A response to
Long-Term Algebra Reform: Democratizing Access to Big Ideas

Picture a student in a mathematics class today—a student whose career goals will be achieved through a technical school or community college, or a student who intends to enter the work force after high school, or a student who wants to become an engineer; a student who has to live and work in a world of information and technology. What algebraic skills and concepts should each of these students know and be able to use? Algebra has always been prominent in mathematics. It has not always been accessible to all, and there have been many recent efforts to address issues raised by technology and equity. In his paper, Long-Term Algebra Reform: Democratizing Access to Big Ideas, presented at the Algebra Initiative Colloquium, Jim Kaput states that "conferences of this sort do not answer questions.... they set agendas."

As the mathematics community begins to set an agenda for algebra in the mid-1990s, there are issues that need to be raised from the perspective of a secondary classroom teacher. The reform of the algebra curriculum is embedded in the reform of the mathematics curriculum; reform efforts in mathematics are moving toward building vertical knowledge instead of grade level knowledge, strands instead of courses. Such reform efforts are suggested for algebra. As a consequence, in the comments made both by Kaput and in what follows, it is often difficult to distinguish between algebra in particular and mathematics in general.

Teacher Preparation

A critical element in any agenda for algebra reform is the mathematical backgrounds of teachers. According to a U.S. Department of Education Survey of Recent College Graduates Transcript Data File, 56.3 percent of the full-time mathematics/science teachers in 1988-89 who obtained bachelor degrees in 1985-86 took at least one calculus course. If the data is representative of classes from other years, it seems to indicate that the mathematical background of 44 percent of mathematics and science teachers is very deficient. Without a knowledge of mathematics, without an understanding of the major themes in the branches of mathematics, most secondary teachers can only have rigid and narrow views of the curriculum, with little understanding of what mathematics is really all about; of how to make connections within mathematics and to other disciplines; of how to apply mathematics to solve rich and meaningful problems. Their lack of knowledge will prohibit them from making choices that might provide their students with a rich and diverse understanding of algebra. Their algebra lessons will be driven by their own remembrances of the course and constrained by their limited knowledge to the pages of their texts.

Strong recommendations are being made that algebra should begin in early grades and, thus, will be taught by the teachers in those grades. According to the survey, only 8 percent of the general education majors took at least one calculus course. This is not as surprising as the data about secondary teachers but does imply that the amount of mathematical preparation for 92 percent of the general education teachers is very limited. Not that it is necessary for all teachers to take calculus, but most courses prior to calculus at the university level are either remedial or duplicate high school courses. Preservice teachers currently have few meaningful options. For most teachers, it is impossible to teach well what you do not know. A seventh grade teacher who does not understand how to write the equation of a line, given two given points, will not do justice to algebraic concepts. A reform agenda must include some way to provide adequate mathematical preparation for those who are to teach algebra.

Tradition

Tradition is part of the problem. Kaput calls for a new, more diverse curriculum that has room to grow and change. Before the change, however, the agenda that is set must pay attention to the current curriculum and pay attention in at least two ways: to the role of algebra as a filter and to the understood content of algebra as perceived by those who took and who teach it. Algebra has been reserved as the high school "entry" course for mathematics, particularly for college-bound students. In many schools, only "good" students are recognized as needing algebra, and schools make judgments on which students are eligible for this gateway course. In addition, current trends in industry and business indicate a demand for workers with an increased ability to handle algebraic concepts in order to function efficiently in the workplace. As algebra reform is undertaken, it is essential that those algebraic concepts necessary for everyone are in a curriculum accessible by all and that algebra as a barrier to college access be removed. Beginning algebra in the early grades may alleviate the problem, but any potential solution must carefully consider all possible implications.

Teachers perpetuate tradition in their classrooms. If teachers are to change what they teach in algebra and how they teach it, they need to be convinced that what they currently do is wrong, or at least not sufficient. Many of them are good at teaching algebra. Their success is validated by students and parents. Students do master procedures and learn to perform well on short-term goals. Most teachers do not have the opportunity to teach those same students several years later to see the long-term results of their efforts, to see that second year algebra is a replay of the first year. With only generalities such as "the role of algebra as a language" as a motivation, with no rationale that makes sense to them, teachers will continue to do as they have done "successfully" in the past. And those who are not successful will emulate those who are, because that is how teaching algebra works and how they are judged. And thousands of students will continue to miss the mathematics, become mystified by symbols and join the chorus of adults who say "I never could do algebra," yet think "doing" algebra is important.

There is an institutional tradition about algebra. The school community and the public have expectations about the content of algebra, and they do not lightly accept changes that conflict with these expectations. What they learned or did not learn is what should be. In addition, tradition has imposed a sequence on the subject that teachers find hard to modify. In nearly every algebra course, the chapters are standard and the exercises similar: graphing appears in the second semester of first year algebra; the quadratic formula comes after factoring, systems of equations are studied separately from equations (usually five chapters apart); functions do not appear until the end of first year algebra and often not until the second year of the course. Further, the essence of algebraic concepts has been lost by dissecting each into minute parts: finding the solution to a system is not the focus of instruction, but rather the use of the substitution method, the addition method, Cramer's rule, the graphing method—to what purpose students are not sure! Tradition and reliance on paper-and-pencil processes have decreed an order and linearity in the approach to algebra that is very difficult to displace, yet as technology makes new investigations possible and opens the doors to new ways of thinking, old questions now occur at very different times. For example, students thinking about linear graphs and rate of change early in algebra often ask what happens when the change is not constant. In rethinking algebra, these old questions and corresponding new ones must be addressed.

Content

Before a solution to the problem of making algebra accessible to all students can be implemented, before technology can be incorporated into algebra in significant ways, the agenda has to contain some direction about content. Teachers need specifics. Why should a topic be included in the mathematics curriculum? As an algebra teacher it is not uncommon in today's classroom for me to stand in front of a class that is about to learn a topic from the standard curriculum and ponder: "Why should I teach this? Of what use is it to students? Is there a better way that will be more functional"? Mathematics should be done for a profit. Unless students see some gain from using mathematics, most, in the long run, merely go through the motions.

Mathematics is important if it

- Makes something easier;
- Generalizes a rule or process that is useful;
- Provides a solution to a problem of interest;
- Is fun;
- Provides a link to understanding something else;
- Is necessary to learn other mathematics.

A quick glance through an algebra text yields many topics that might fail to meet any of the above criteria, particularly for the sequence and position within the mathematics curriculum in which they are located. And even more significantly, the message about importance can not be one that is apparent only to the curriculum designer. If students do not perceive the gain from using algebra they will cheerfully (or not so cheerfully) continue to do as they have in the past, avoid the subject, fail the subject, or tolerate the subject mastering superficial knowledge in order to survive.

Furthermore, if calculus is not the capstone course, a reasonable option advanced by Kaput, what is? What mathematics goals will drive the K-12 curriculum? What targets should teachers aim for? Without clear mathematical outcomes and expectations, teachers can not effectively design curriculum nor teach students. In today's classroom, it is easy:

nearly every concept is driven by its anticipated use in calculus. The entire first year algebra course sets the stage for calculus. Kaput suggests bringing new topics into the core curriculum, topics such as matrices, statistics and uncertainty. To what end should these be taught and will those at the university level who receive the product from such a curriculum value students with new and different mathematical knowledge?

Assumptions

By far the most significant observation that can be made from my experience in the classroom is that my perception of what students heard me say, of what they know and how they approach a problem, is, at the very least, inadequate and usually dead wrong for most students. They bring background knowledge of some kind—which I often ignored in the rush to tell them everything I want them to know. The ways in which students construct their own understandings of algebra are not always obvious, particularly when they have had access to graphing calculators for most of their work in mathematics. My way of reasoning and approaching a problem is not theirs; and they are putting pieces together in ways I cannot comprehend because I bring a traditional background to working with technology and to solving problems. I have a solution and a technique for finding that solution. It is difficult for me to reconstruct the thinking process I used before I had the answers. It is even more difficult to try to reconstruct the process a student immersed in technology might use. An agenda for algebra for the next decade must include helping teachers learn to listen to students and to act on what they hear to promote the development of mathematical knowledge.

Research and its contribution to increased understanding about how students learn can play an important role in thinking about algebra. While there are many issues, such as the relation between practice and context, that need to be addressed, there is a growing body of knowledge that should help teachers understand how students learn algebra and what they learn when they learn. A critical factor for a reform agenda is to provide effective ways to disseminate the results of such research to teachers.

Symbol sense is a critical issue, but in new ways as well as old. Students need to use symbols to communicate with technology. What is 7x on the calculator and how does 7x2 differ from (7x)2? Do students actually see the relation between tables, graphs and symbols? What do they see when they look at a table? How do they move from a discrete set of elements in which a pattern may or may not be obvious to a continuous graphical representation? What characteristics of an equation make sense when the equation is used to describe a table of values or a graph? Are the points on a line only the ones that you put there from the table? What do students think when they see a quadratic expression, a quadratic equation, a quadratic function? Students do not recognize the equivalency of \$50 + 2x and 2x + \$50, probably because they have entered the misty land of symbols where what you see is not necessarily the same from place to place in the course. What are the variables in $y = mx+b$? Students make brilliant conclusions that represent deep understandings yet often don't recognize them the next day written in "official" algebraic language.

Structure

Teachers must have a sense of the big picture of mathematics. Cross-cutting concepts such as equivalence and its relation to re-expression, transformations and the search for invariance, symmetry and its role in analysis need to be made explicit in the organization of algebra. Kaput suggested central concepts in a very different way: sensitivity, orbit, closeness, product and quotient structure, sameness. In rethinking what is important in algebra, whatever organizational structure is used, that structure has to be apparent or teachers have no goals on which to focus lessons. Algebra thought of as representation as Kaput suggests might allow students to abstract from the process to the concept. For example, in current algebra manipulations, the power of equivalent forms to render "quantitative relationships more easily recognizable" (Kaput) is lost in the process of creating the forms (whether this is done by technology or by paper and pencil; only a few recognize what is actually taking place.) In addition, the tendency to redefine all of algebra as "function," indeed as input-output, would seem to limit the usefulness and power of algebraic representations. Examples from physics such as $F = F1 + F2$, a net force composed of two forces, or the use of algebraic representation for geometric objects seem to reinforce the need for students to have access to different ways to think about representations (and meet one of the criteria above for important mathematics).

Finally, as algebraic content and structure are examined, there seems to be a danger that some will construct a "new" algebra using technology to do the same things tomorrow that can be done by paper and pencil today. This may be one reason for the current trend, which seems to emphasize function as the major focus of algebra, rather than one of several very significant concepts. Technology will continue to evolve and, as

Kaput's examples suggest, new visions of what can be done with technology will demand even further modifications of the curriculum. The impact of this use of technology on teaching and learning and its integration into the curriculum is not yet even conceived, but surely must be accounted for in any reform.

The Big Picture

The overarching curricular goals, democratizing access to big ideas, and the enabling goals Kaput laid out in his paper capture the essence of what the agenda for reform must consider. Attention must be paid to diverse interests of students as they enter the later years of secondary schools. And articulation must carry out the recommendations, and unless teachers are partners in the change process, informed about the agenda, knowledgeable about the direction of the mathematics and convinced that the reform will benefit their students, algebra reform will be just another phase...management by objective, outcome-based education, mastery learning...and teachers will close their doors and do what they have always done for "this too will pass".

What Is the Appropriate K-12 Algebra Experience for Various Students?

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A response to
Long-Term Algebra Reform: Democratizing Access to Big Ideas

While this question has been much debated forever, but especially over the past 10-15 years, my guess is that in one sense we really are in quite substantial agreement about the answer. Modulo some disputes over fine points, I suspect that all of us want students to acquire the ability to represent and reason about patterns relating quantitative variables (and quantitative relations that arise in numerical modeling of visual patterns) and the ability to use symbolic expressions to describe and reason about patterns and operations of more general sorts.

We probably also agree on many basic principles of how students can most effectively acquire that understanding and skill and on instructional principles that follow from them. We have already produced some very impressive demonstrations of how technology can be used to enhance instruction and extend the problem solving power of our students, and Jim Kaput's paper stretches the imagination even further. Quite a few people have thought quite carefully about how the algebra strand of school mathematics can be better formulated to prepare people for effective quantitative reasoning in technical work and the judgments that face each of us who want to be involved in the social/political decisions that shape our futures. Furthermore, the visions of algebraic knowledge that are emerging from those considerations emphasize general mathematical understandings and skills that, as best we can know now, should be adaptable for lifelong mathematical growth.

The progressive choir in mathematics education sings nearly as one voice the lyrics of variables, functions, relations, modeling, multiple representations, applications, problemsolving, and so on. Unfortunately, the beauty and wisdom of our message very often falls on unappreciative or skeptical ears. Why is this the case and what is to be done about it? In what ways has Jim Kaput's plenary analysis been insightful and what are the issues that are not so satisfactorily dealt with?

It has always seemed to me that we have long had the right ultimate algebraic goals for our students, but the path to those goals was so long and arduous that very few ever acquired meaningful payoff from their labors. The truly miraculous impact of technology was to open up our imaginations to entirely new and more efficient paths to algebraic power—by increasing the potential effectiveness and pace of teaching and learning and by making some legs of the traditional paths less critical than we had previously thought. We've developed those dreams into curricular proposals and even produced a few fledgling working school programs. But the world isn't beating a path to our doorways.

In some sense this failure of most schools and the concerned public to adopt our ideas is a deep puzzle to me. In the face of overwhelming evidence that our current curricula and teaching are not successful by any criteria, the public and many teachers cling ever more tenaciously to traditional goals and practices. On the other hand, we have to be candid and admit that we have precious little convincing evidence that our ideas can be converted into working realities.

Has Jim Kaput set the right set of goals for our development and demonstration efforts aimed at winning acceptance of new and powerful approaches to algebra in school? I'm not sure. To pick on just a few of his many provocative proposals:

- **Replace the layer cake with interwoven strands.** This is a very attractive suggestion, based on the plausible premise that a curriculum of interwoven strands will help students develop the kind of connected knowledge that is easier to retain and to deploy in realistic problem contexts outside the formal mathematics classroom. But, as desirable as this proposal seems, it doesn't seem critical to the improvement of algebra. As with so many other reform ideas in the air, the efficacy of integrated curricula has not yet been demonstrated.
- **Integrate the basic ideas of algebra into K-8 mathematics.** Again, this suggestion has real promise. There are other countries in the world where algebraic ideas are standard parts of elementary school curricula. However, there is a big difference between introducing some ideas of algebra and accomplishing a substantial treatment of the major concepts and skills of that core subject. While I'd be the last one wanting to be quoted as saying "kids can't," there seem some genuine

cognitive obstacles to acquiring a useful understanding of algebraic methods. We need to know a lot more about the potential for accelerating the algebra agenda in curricula (demonstrations in realistic classes with typical teachers and students) before we get very excited about this notion.

In many of the newer proposals to exploit technology in revision of algebra, we are actually calling for students to demonstrate more sophisticated conceptual and macro-level thinking than the absorption of rote symbol manipulation routines require. For this reason, and the practical consideration of teacher capability to deliver thoughtful high-level conceptual mathematics instruction in the early grades, prospects of moving algebra down to the grades seem problematic.

- **Enrich the core 9-10 curriculum.** It goes without saying that most current school curricula, especially in algebra, leave students with an impoverished and limited vision and understanding of the subject. However, proposals to enrich current curricula don't really promise help with development of basic ideas of algebra. If anything, it seems that the more plausible proposal for American mathematics education is to focus attention on a smaller number of major, powerful ideas and to develop student ability to deploy those ideas with flexibility and imagination—not to load more topics into an already busy, but often trivialized, curriculum.

Whether you agree or disagree with any or all of Jim's suggestions, it seems to me that there is one glaring omission in his remarks. All of us who want to change the world—to win friends and influence people to a new way of thinking about algebra and its teaching in school mathematics—have a first challenge to construct existence proofs of our ideas. As plausible as any of the current proposals for algebra might be, we are obliged to demonstrate that teachers can be prepared to use the new curricula effectively in schools. The history of mathematics education is littered with proposals that had compelling theoretical rationales but could not be effectively implemented, and many of the current reform proposals have far too little supporting evidence to justify broad implementation.

Some of the required demonstration work is under way, and we should be trying to get our best new ideas woven into the experimental curricula being developed and tested. We should also be testing radical ideas on the fringe of what is feasible, without putting kids at too great a risk. It's certainly true that it often seems as if things couldn't get a whole lot worse than they are right now. However, a responsible scientific attitude compels us to accumulate solid research and development evidence before we expect schools and the public they serve to adopt our proposals. Until we have that evidence, we won't really know what the appropriate algebra experience is for various students.

Algebra at the College Level

Michael Artin
Massachusetts Institute of Technology

NOTE: This is an edited transcript of my informal talk at the Algebra Initiative Colloquium. It has three parts:

- What is algebra, and why is it important?
- A schematic description of an algebra course.
- Some thoughts on the algebra curriculum.

What Is Algebra, And Why Is It Important?

It seems reasonable to examine these questions before discussing curriculum. Though there are fairly satisfactory answers to the first question, the second one remains a mystery to me.

Here are three attempts at a description of algebra: Algebra is

- The language of mathematics;
- Working with x; and
- The study of the algebraic operations +, x and their analogues.

I like the first description, the one Carole Lacampagne used in her announcement for this workshop: Algebra is the language of mathematics. Algebra is one of the most abstract parts of mathematics, and I've always felt that language and abstraction are closely linked. In mathematics, when the right definition is made, an understanding of the abstract concept evolves out of it. But when I mentioned this feeling to a neurologist a number of years ago, he said:

> No, that's wrong. The power of abstraction is much more shallowly rooted in the brain than language. People with a brain injury often lose the ability to think abstractly, though the language capability remains.

This was interesting, and so I asked him: "How do you test abstract thinking?" He replied, "Oh, we have standard tests. For instance, why is an apple like an orange?" The thing is, I flunked the test. I thought: "Well, they're both sort of round." But the apple has those dents, so I rejected that answer. Then an orange is orange and an apple might be red. It became painfully clear that one of us was in the wrong field. After a while, he put me out of my misery. He told me that the right answer is "fruit," and we changed the subject.

During my college days I had some summer jobs doing manual labor, and I have a sad recollection from that time which also makes me wonder in what sense abstract thinking is fragile. It is about a mentally handicapped man on one work crew who followed the Brooklyn Dodgers. Though it was near the limit of his ability, this man always learned the result of the game before coming to work. He would start the morning by announcing the score several times in a loud voice. "Dodgers 5, Giants 3 yesterday." The rest of the crew usually responded gently. Then as the day went on, he would ruminate on his one piece of information, working out its implications and reporting his conclusions to us from time to time: "Dodgers beat the Giants",...,"Giants lost," and so on. Each reformulation gave him the pleasure of a new insight, and I found this so remarkable that I remember it clearly today. Had it not been for his birth injury, he might have become a mathematician. So, though there is clinical evidence to support the neurologists' view, I'm not completely convinced.

"Working with x" is roughly the definition that Victor Katz gave in his talk:

> The science which employs letters in reasoning about numbers, either to discover their general properties or to find the values of an unknown number from its relationship to known numbers.

This description is still applicable to most of high school algebra, but for algebra at the college level, often called abstract algebra, it isn't completely sufficient. Hence our third definition. It is very close to the second description, but its allusion to analogues allows us to fill in something new.

The seed of the third description is already in the opening line of Euler's 18th century algebra book. Here is a rough translation, but the original is nicer:

> Everything which can be multiplied or divided, to which something can be added or from which something can be taken away, will be called a quantity.

But up to 1900, texts such as Weber (university texts) used the "working with x" definition, and since then, people haven't really tried to define the term "algebra" at all.

Finding a satisfactory answer to the second question: "Why is algebra important?" is much harder. From some points of view, the importance is clear. One may mention the historical importance of the Greek "quadrivium":

arithmetic geometry
astronomy music,

and the modern trinity of core mathematics:

geometry
algebra analysis.

Though algebra is an Arabic word which dates from the middle ages, the subject has been a large branch of mathematics since the time of the Greeks.

Now I have two puzzles to which I want to draw to your attention. I think that a satisfying explanation of why algebra is important ought to lead to an understanding of these observations:

- In concrete examples, the algebraic laws seem to be superimposed artificially on structures which occur naturally.
- Algebraists study, almost exclusively, binary laws of composition.

Here is what I mean by artificiality of the algebraic laws. Let's take three fundamental examples of algebraic structure—the integers, the group of symmetries of a figure, and the state space—and ask: What are the algebraic laws?

The integers are for counting: 1, 2, 3,.... That's the integers. What are plus and times? These laws are superimposed on the counting structure. Now it is quite hard to think back to the integers, but I tried, and I came to the conclusion that the algebraic laws are artificial. Certainly addition and multiplication are useful because they help you to count. You learn to manipulate them and they are useful tools. And of course they can be characterized axiomatically. I'm not satisfied.

Take symmetry. Say you have a regular polygon. You can rotate through some angle, or you can flip it around an axis of symmetry. What do mathematicians study? They study what happens if you first rotate and then flip—the composition of those symmetries. Why is that important? This is really unclear to me.

Finally, take the state space. In physics, you make some measurements on a particle and you assemble them into a "state" vector. What does it mean to add two states? A Volkswagen Beetle is traveling down the turnpike with a lady bug crawling across the back seat. An observer at rest computes the state of the Volkswagen, and a passenger computes the state of the lady bug. When you add the two you get the state of the lady bug, viewed from the position at rest. That's superposition of states, and it is quite forced. So the algebraic laws are artificial here too.

Also, the second point: Why do we always study binary operations? You know us algebraists, we've been at it for a long time. We've written thousands of pages. And of course there are laws that aren't binary, but nobody studies those. That is another puzzle.

There is a striking contrast here between algebra and geometry. The utility of geometry is quite a bit less clear than that of algebra, because you can't use geometry to compute. Nevertheless, its place in the trinity seems secure. I think the reason is that geometry is explained cognitively: It is the way our visual cortex expresses mathematics. I'd like a similar explanation for algebra, one which clarifies its relations to other cognitive functions such as language. Unfortunately, such an explanation hasn't yet been found.

"God made the integers; all the rest is Man's work." This is a famous quotation of Kronecker, but I don't believe it. In spite of what seems to be artifice, there has to be a natural explanation of "the rest," because if Kronecker were right, then "the rest" wouldn't be very important. But algebra *is* important. It is one of the holy trinity of mathematics. So algebra must have a cognitive explanation too.

A Schematic Description of an Algebra Course

How do you discuss curriculum? The only really sensible way is in the context of a particular course. So I'm a bit hesitant to show this. My description is a three-dimensional array: objectives, content, and structure. Ideally, I'd like it to be completely tautological. I made it up in consultation with several colleagues because I thought it might be useful at this workshop, but I'm sure you will want to amend it. If you wonder about the order, it is alphabetical in each column.

Objectives	**Content**	**Structure**
Application	Context	Class
Engagement	Examples	Exercises
Logical Reasoning	Theory	Tests
Technique	Window Dressing	Text
Understanding		

Let me say something about what I mean by the words:

- By **application** I mean the ability to take the material learned and to apply it to a new situation. That is certainly an objective.

- **Engagement** means engagement of the students. I mean arousing their interest: motivating them for the course, and beyond. Of course, there is another engagement that is very important: that of the teacher. But that is not an objective. (Our transit system seems to have two objectives: the convenience of the commuters and the convenience of the employees.)
- **Logical reasoning** is supposed to be a catch-all phrase: the ability to abstract and conceptualize as well as to make logical arguments which require several steps. Things like that. If you ask people who have taken mathematics courses in college and have gone on to something else in life what they got out of studying math, that will be what they say: The ability to do such things.
- **Technique** means facility in making the computations that are required for the particular subject.
- **Understanding** refers to the content of the course.
- **Context** places the material of the course in a broader framework. It might be applications, or history.
- By **examples** I mean conceptual examples. I don't mean to solve $x + 3 = 5$, but addition of integers, or the group of symmetries of a figure. What are the basic structures that one is talking about? That is what one should think of when talking about examples in an algebra course at the college level.
- **Theory** is the synthesis of the examples.
- **Window** dressing is extraneous material that is not properly attributable to context. It sounds like a pejorative term. Actually, window dressing is quite important.

Some Thoughts on the Algebra Curriculum

Should we revise the curriculum? I ask this as a rhetorical question. I think that we should, and I see two reasons why it it is important to do so.

One reason is that the students coming in are changing in many ways: demographics, TV, computers, and so on. Since these changes have been discussed frequently, it seems unnecessary to talk about them here. And as the school curriculum is revised, that will have an influence that we will need to take into account at the college level.

Another reason is that people at major universities around the country often tell me they hate to teach the algebra course. They say, "It is a bunch of definitions and some abstract lemmas; nothing serious gets done," or else "The students are so lousy, we can't teach them anything." I have a hard time understanding these depressing remarks because I've taught algebra for many years and, if anything, I enjoy it more every year.

Here is a caricature of the life cycle of a curriculum. I've made it on the basis of a book because, traditionally, the text has played a major role in defining curriculum.

Phase 1: Influential book appears and is adopted.

Phase 2: Books pare down in order to cover more.

Phase 3: Books pare down in order to become accessible.

Phase 4: Course is reduced to a skeleton.

In an algebra course, the influential book might be Birkhoff & MacLane, a great book which appeared 50 years ago. That's Phase 1. People adopt it, and curriculum is established.

Then comes Phase 2: Books pare down in order to cover more. Why? Because, the influential book defines the syllabus. This means that changes are treated as additions. The author of the next text may want to get to certain interesting points: to cover another topic, or to treat something in more depth. It might be the Sylow Theorems; they aren't in Birkhoff & MacLane. The existing text has to be compacted to make room for the new material. So there are natural forces that result in paring down of the books.

Phase 3: Books pare down in order to become accessible. Phase 2 books are too hard because they are pared down and include more. Other forces are also at work. In the beginning, Birkhoff & MacLane would be taught primarily to senior math majors. Then it became junior majors, and then some sophomores and nonmajors. You teach a wider group of students at earlier stages in their education.

Finally, in Phase 4, the course has degenerated. This is the current status of some algebra courses.

I don't really believe this caricature. It is Shiva worship. Many books have been written since Birkhoff & MacLane, and they have made positive contributions. Brahma and Vishnu are also at work. But the faculty views that I quoted above lead me to think that this has taken place to some extent.

Since books set the curriculum, we should pay more attention to textbook writing. Look at the following table from the 1990 NRC report "A Challenge of Numbers."

TABLE 5.3 Professional activities of four-year college and university mathematical sciences faculty

	Mathematics			Statistics
	Universities	Public four-year colleges	Private four-year colleges	Universities
Classroom teaching performance	70 (3)	81 (2)	96 (4)	71 (6)
Published research	96 (0)	70 (10)	26 (39)	100 (0)
Service to department, college, or university	31 (5)	63 (5)	66 (0)	31 (11)
Talks at professional meetings	42 (5)	49 (11)	13 (28)	25 (11)
Activities in professional societies or public service	22 (8)	45 (4)	33 (9)	31 (6)
Supervision of graduate students	34 (7)	21 (32)	—	81 (0)
Undergraduate/graduate advising	9 (22)	24 (20)	39 (12)	21 (21)
Expository and/or popular articles	22 (13)	37 (14)	14 (40)	14 (19)
Textbook writing	9 (35)	17 (35)	11 (58)	12 (50)

This chart reports on a survey of department heads, who were asked what is important and what is unimportant for promotion and tenure of their faculty. There are several interesting things in the table, but I want to draw your attention to one in particular. Only 10 percent of the chairs think that it is important for their faculty to be engaged in writing texts. I've been trying to understand this remarkable statistic, and I hope it will change.

Let me show you my schematic description again, listed this time in what seems to me is the order of importance:

Objectives	**Content**	**Structure**
Engagement	Examples	Exercises
Application	Context	Class
Logical Reasoning	Theory	Text
Understanding	Window Dressing	Tests
Technique		

You will probably agree that engagement is the most important objective of a course. Though all of the objectives are important, technique comes in a weak last.

Examples and context are more important than theory. Window dressing is important, but it is not as important as the other parts of the content.

Exercises form the most important part of the structure of the course. Class and text are perhaps equally important.

Notice that, though we're thinking of this as an algebra course, our description has nothing to do with mathematics, with the possible exception of logical reasoning.

General Principles About Curriculum at the Undergraduate Level

All Topics Should Be Important. This sounds tautological but it is not always followed. We should be ruthless in asking: "Is it important for the average student in the class to learn this material?" If not, throw it out. There's plenty that is important.

I see three criteria to apply in assessing the importance of a particular topic: beauty, historical significance, and utility. For example, the impossibility of trisecting the angle with ruler and compass is a topic which ranks high in beauty. It deserves a reasonable mark for historical significance, but it fails the utility criterion completely. One might well ask: Is it worth doing in a college algebra course? Well, I often discuss it. Should it be locked into the curriculum? I don't think so.

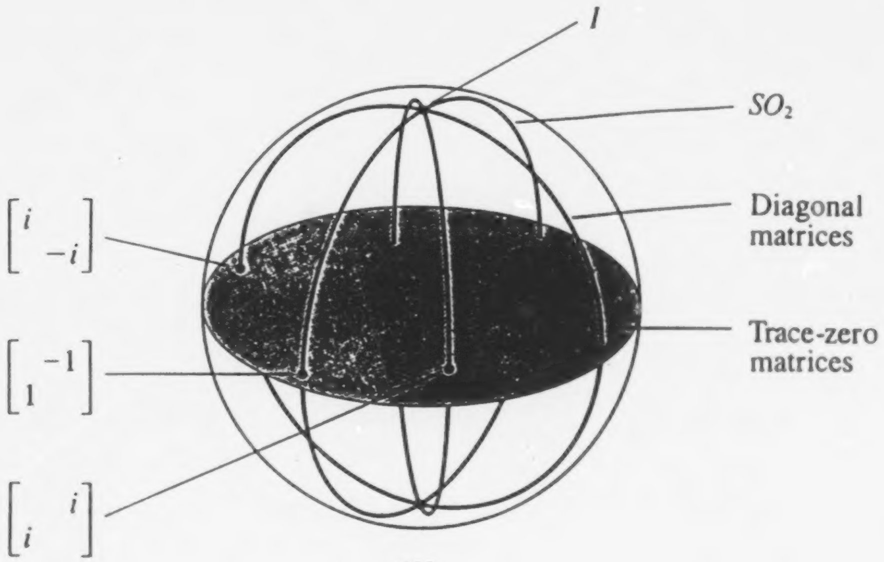
An admonition for this workshop: Don't lock the syllabus in place. Let's not tell people exactly what belongs in the course. For an undergraduate course, the most important thing the students should come out with is a familiarity with some examples—some basic structures on which they can build their understanding. That is more important than theory. In a graduate course one is training professional mathematicians, and one needs to develop the theory systematically.

Appropriate examples should precede the introduction of new concepts, and in general, special cases should come before the general. (I'm breaking this rule today.)

Minimize Direct Attention To Service. By this I mean covering a topic in your course only because another course needs it—teaching exterior algebra because it is useful in differential geometry, for instance. Of course service is important, but I'm not in favor of such direct attention.

Minimize Problems Which Are Pure Drill. This is justified because technique is at the bottom of the list of objectives.

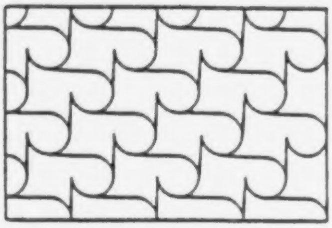
Now I'm going to talk a bit about window dressing. It came last on the list, but I still want to encourage people to pay attention to it. Please excuse me for trotting out two examples from my own algebra course.


I
SO_2
Diagonal matrices
Trace-zero matrices
$\begin{bmatrix} i & \\ & -i \end{bmatrix}$
$\begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$
$\begin{bmatrix} & i \\ i & \end{bmatrix}$

SU_2

This is a picture of the special unitary group SU2, which is a three-dimensional sphere. For years, my students complained when I tried to get them to think about this important example. Then one day I had a brainstorm: longitudes and latitudes. It turns out that the longitudinal great circles have algebraic significance. They are subgroups. And if you take a horizontal slice, then, because the group is three-dimensional, you get an ordinary two-dimensional sphere. These latitude spheres are the conjugacy classes. My students' attitude changed remarkably once I organized the material along these lines. They began asking for more reading. But of course, calling them longitudes and latitudes is window dressing.

Following current fashion, I'm encouraging the students to work in groups on their assignments. This has required a major change in my attitude. I think it is good for social reasons, but also because it allows me to assign more extensive problems. Here is the latest version of a problem which I've assigned regularly for perhaps twenty years. I'm still working on it.



1. Find a way to compute the area of one of the hippos which make up the pattern above. Do the same for one of the fleurs-de-lys.
2. Find two different fundamental domains for each of the patterns.
3. Prove that if D and D′ are fundamental domains for the same pattern, then D can be cut into finitely many pieces and reassembled to form D′.
4. Find a formula relating the area of a fundamental domain and the order of the point group of the pattern.

The first problem is the hook, and of course calling them hippos is window dressing. (The students claim these are not hippopotamuses. I say that they are hippo heads, and the students accept that.) As it turns out, you can cut up one hippo and reassemble the pieces to form the parallelogram which joins the tips of four ears. So the area of the parallelogram is the answer. That is what I hope students will discover. You can do a similar thing with the fleurs-de-lys.

If you move the hippo around the plane by all of the symmetries that the pattern has, you will cover the plane without overlaps. This means that one hippo makes a fundamental domain. The parallelogram is also a fundamental domain for the hippo pattern, and half a fleur-de-lys is a fundamental domain for the second pattern.

The last problem is quite challenging. The order of the point group counts the angles of rotational symmetry and the directions of the axes of reflection. Since the order is dimensionless, there is a missing term.

I think this is a good format for an extensive problem. It has an attractive hook and suitable window dressing, it is in increasing order of difficulty, and it doesn't give all of the answers away. My experience is that though the problem is challenging, the students can get quite far on it, especially working in a group. It would embarrass me to show you the condition the problem was in when I handed it out just 5 years ago.

Creating exercises is hard work, and good ones should enter the public domain, because they're so difficult to make. That is why the same problem from China from 200 B.C. is still in the curriculum (see Katz, this volume). We need to find ways to encourage people to develop thoughtful exercises and to facilitate their dissemination. These days, exercises for a course are most often taken from the text. So, the exercises form the most important part of the book. But assembling them is usually the last thing an author does.

A question for the workshop: What mechanisms can the community introduce to stimulate an improvement in teaching and in solving the pedagogical problems? The pedagogical problems don't really change much. We have been teaching modern algebra to undergraduates in this country for 50 years—since Birkhoff and MacLane appeared—but the difficulties one encounters when teaching bilinear forms, for instance, have not been solved.

Let me return for a moment to logical reasoning and abstraction. I want to make the point that these skills are useful everywhere, not only in mathematics, and also that they are largely learned. My college roommate, an archeologist, came to visit last spring, and while we were talking about mathematics he said: "We have no abstraction in archaeology." Actually, archaeology is about as abstract a subject as I can imagine. If anyone is building castles in the air, literally, it is the archeologist. It happened that earlier in the day he had told me that he was teaching about the "Hunter/Gatherer" in his course. But he was completely unconscious of the fact that this is an abstract concept.

In the early thirties, the cognitive psychologist Alexander Luria interviewed people in the Kazakh region of the Soviet Union who had no formal education whatsoever. He discovered some very interesting things about their treatment of abstraction. One experiment consisted in giving people several pieces of yarn and asking them to sort the yarn into groups. People who had even one year of school education would sort by color: blues together, and so on. But those with no formal education tried to sort according to the uses that a particular yarn might have. Luria also played a syllogism game:

Question: "In the North, where there is always snow, the bears are white. Novaya Zemlya is in the far North and there is always snow there. What color are the bears?"

Answer: "I've never been to Novaya Zemlya," or "The only bears I've seen are black."

Luria's report contains many such experiments. It makes good reading, but it went unpublished for 40 years because the Soviet authorities thought that it denigrated the peasant.

So you see, logical reasoning and abstraction are learned skills. Given enough motivation, anyone can make progress in learning them. For example, they aren't going to catch me with that fruit question again! And if we expect our students to learn these things, then we have to teach them.

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Algebra Initiative

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I think Mike Artin has made some very good points, outlined the areas that we should be concerned about and given us many thought-provoking ideas about our undergraduate curriculum. Here are some reactions.

I appreciated the discussion of what algebra is, its historic origin and central place in mathematics. There is much food for thought here. I am uneasy about tracing its origin in cognitive processes since I think that would imply a certain universality and, as I understand it, our mathematics is the result of philosophical attitudes in Western Civilization (from ancient Greece). What type of algebra developed in Africa or Asia? I do not regard this as a criticism as we do have difficulty teaching algebra, and these cognitive concepts might help. I think most mathematicians regard certain topics as belonging to algebra (i.e., linear algebra, group theory, ring theory, fields, number theory). There are new areas whose relation to algebra is not so clear to me, combinatorics, for example. Some parts of combinatorics seem to fit these definitions of algebra, however, other areas of combinatorics are not so clearly algebraic.

With its central place in mathematics, to me algebra needs no justification; although its great usefulness has been mentioned several times, in physics for example and more recently in cryptography and coding (as mentioned by Susan Montgomery).

I liked Artin's division of a course into objective, structure, and content. I agreed with everything he said here, but I have a few comments. I particularly liked the goal of getting the students engaged or involved. I think this is very important and is something we should concentrate on. I would like to add to this list, motivation. It is one way to get students engaged. It is a challenge to determine what motivates students. Sometimes applications do, sometimes not. Certainly, there are other motivations.

I agree that part of our job is to teach abstractions and logical reasoning. I think we should go into this in more detail. I assume logical reasoning is the same as the proofs which so many members of our panel have raised as the area where students have the most difficulty. In addition to finite induction, we can teach other proof methods as mentioned in Professor Artin's fine algebra book. These others are proof by contradiction and proof by divide and conquer (dichotomy). I would add to these proof by construction. These proof techniques can be mentioned explicitly and also pointed out in the course of explaining the material. Related to this, I think we should teach students how to communicate about the mathematics which they are learning both orally and in writing. This is not easy. None of this is easy. We also have to consider how to teach abstractions. It seems to me this can only be accomplished by many concrete examples and then pointing out their common features.

It is certainly better that examples precede any theory. I agree that exercises are important, and I would add that computations should have an important place here. However, exercises should be graded or looked at in some way. There is a problem here. We do not get graders for most of our courses and grading can take a lot of time.

Clearly, we should eliminate pure drill and emphasize important topics, but there are bound to be disagreements about what these are. I like window dressing. Anything that humanizes our mathematics should be added—personal stories, anecdotes, history. I would like to briefly describe our department and our algebra courses in order to describe more fully what Professor Artin has called the demoralization of the faculty and to demonstrate that there is a need for reform.

We have a faculty of approximately 70 people in diverse areas, pure and applied mathematics, computer science, statistics, and mathematics education. Our undergraduate algebra consists of linear algebra in two flavors, two semesters of pure linear algebra and one semester of applied linear algebra. There is also a semester of abstract algebra. These algebra courses are usually begun in either the junior or senior year. The mathematics education majors take the pure courses. The applied students, computer science majors and engineering students take the applied course. As far as I can see, the difference between the courses is not a matter of applications. The pure course stresses vector spaces and linear transformations. There are more proofs in the pure course and more techniques and examples in the applied course. There are five sections of the applied course running this semester and one section of the pure course. I interviewed 12 faculty members who taught these courses at sometime (including myself). The estimates (of the instructors) of the percent of students in the pure class who understood the course ranged from about 20 percent to 50 percent. The same question for the applied course elicited the numbers 30 percent to 75 percent. This response is not just due to the fact that the students cannot do proofs although that was certainly mentioned, but to the fact that they did not know what math was all about; in fact, do not understand why a proof is needed. Some instructors mentioned that the students did not have a background in finite induction, set theory, and quantifiers. However, these students were considered intelligent and hard working.

The results from the abstract algebra class were even more depressing. The students were just turned off. This course has 11 students this semester who are working hard and not comprehending. I do not consider this situation acceptable. In general, the instructors liked their texts, however, there is disagreement as to whether the texts are suitable for the students with the backgrounds our students have. These students are not engaged. I think we should consider how to engage them and if algebra is to be part of their education, I think it should begin at a lower lever, probably with more concrete courses.

There is also a number theory course taught and two semesters of undergraduate combinatorics. The feeling of malaise is missing in the combinatorics class. The students enjoy it, find the many applications as adequate motivation and appear to understand. There are some proofs here, but mainly the course involves techniques. There are many, many concrete examples.

There is also a coding theory for upper level undergraduates and graduate students. It runs every year because of demand. The only prerequisite is linear algebra. Many of the topics in the abstract algebra course are needed but they are taught in the coding class. These include a bit on groups, mainly as symmetry groups, rings, and much of the finite fields. Examples and context come before theory. Furthermore, all of the algebraic areas are motivated by their importance to the coding theory. The approach is algorithmic and constructive. There are many, many concrete examples and much computation.

I do not think the experience of our department in undergraduate algebra is unique. Our students are not engaged. They are not getting the enjoyment out of algebra that they should. Many of our algebra teachers are demoralized. I think there is a need for reform.

I agree that in addition to the problems of student engagement, methods of instruction, and course content; there are the questions of teacher engagement, good textbook writing and propagation of appropriate and challenging exercises. In this context, I think it should be noted that excellent teachers have their own, different teaching styles, and that we should not be locked into either a particular teaching style or a particular syllabus. These are many topics for future discussions.

Algebra and the Technical Workforce

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I should like to begin this discussion, in a rather broad setting, with the following question: Why does society give us so much time to teach mathematics? We are one of a very small number of subjects which typically get time every year of schooling. Why? What are the purposes of teaching mathematics to all students? There are answers which do not get at the question—"to prepare for the exams," or "to fulfil the prerequisites for the next mathematics course," things like that. I think that we have four basic needs in mind: The mathematics for practical everyday life, the mathematics for intelligent citizenship, the mathematics for future employment, and mathematics as a part of overall human culture. I don't for a moment wish to pretend that these are disjointed. Many areas of mathematics are highly visible in most, if not all, of these areas. This is an important point, because it means you really don't have to create special sections, or even courses, for the needs of employers. We will see that what employers need most from school mathematics is, to a large extent, what future citizens need for everyday life and to be intelligent members of their communities, and what they should receive as a part of their cultural heritage.

These aims of mathematics education have always been there. True, the relative emphases keep changing: Thirty years ago, we had more emphasis on the culture of mathematics; 10 years ago, we began to see more of the richness of the use of mathematics in citizenship; 5 years ago, we started to highlight employers. But what has driven the need to rethink mathematics education in a fundamental way have been the changes all around mathematics education: technology, applications, and mathematics itself are all very different; and they all change what we teach and how we teach it.

What do employers need from their employees' mathematical education?

- The ability and the instinct to set up a problem.
- The ability to tell whether an exact or an approximate answer is needed, how much accuracy might be wanted, and about how big the answer should be.

- Once these decisions are made, the ability to get an answer by *any* convenient means.
- The ability to work with other people.
- The knowledge that lots of different kinds of mathematics exist and have their uses. The mathematical formulation of a real-world situation may turn out to be probabilistic, or it may turn out to be data driven. (The data driven often involves statistical process control.) The problem may be one of optimization and planning. It may turn out to require a systems approach. It may best be thought of as discrete, or algorithmic. A good summary of the employers' point of view may be found in *Mathematics Education: Wellspring of U.S. Industrial Strength*, published by MSEB in 1989.

Now what does all this imply about mathematics education? What follows from such employer needs? Here are some important aspects of mathematics education from the point of view of future employment.

- Problem finding as well as problem solving. This realization is a key difference between the agenda for action and the standards.
- Emphases on estimation and on the meaning of accuracy.
- The ability to solve a problem mentally, or by paper and pencil, or on the calculator, or on the computer, *and* some experience in making a good choice among these alternatives. Corresponding to the number sense implied by the need to make these choices in an arithmetic context is symbol sense in algebra and beyond.
- Knowing that a large variety of mathematics exists, and the kinds of things it can do;
- Understanding when, how, and why mathematics works.
- Experience with cooperative and group learning of mathematics.
- Above all, experience with modeling real-world situations mathematically, and being comfortable with the open-endedness of such modeling.

In recent years, I have taught several courses in mathematical modeling to mostly in-service teachers at Teachers College. We take situations from human activities, or engineering, or physical science, model them, and study the models and their consequences. What the students have pointed out to me is that modeling, as we practice it, assumes command of the fundamentals of all of high school mathematics—and in some cases, calculus. In a course on modeling, you want to concentrate on the process of modeling, and on the interplay between the simplifications and the results. You use whatever you need in the way of algebraic technique, graphing calculators, computer programs, and simulations. This reminds us of the fact that the organization of high school mathematics into a year of algebra followed by a year of geometry followed by another year of algebra followed by ? is not likely to be the outcome of zero-based optimizing. I think we do it this way in year N because it was done this way in year N-1. This statement holds for all N. The "New Math" of 35 years ago wanted to interrupt this induction, but we were told that the mobility of the American school population precluded such a radical change. One of the most helpful signs from the point of view of employment is that this is no longer an unalterable hypothesis of school mathematics rethinking. A notable leader like Zal Usiskin, for example, is willing greatly to enrich his "algebra" course, as long as he can still call it algebra.

Back to the subject of modeling. Modeling as a central theme for a school mathematics program is a new idea, about which I will shortly say more. It represents a major employer need, as well as an essential aspect of mathematics itself. But the influence of employer needs on mathematics curriculum and pedagogy is not so new. A historical vignette of employer influence on mathematics education: One of my prized mathematical possessions is a report of the American Mathematical Society's Committee on the Teaching of Mathematics to Engineers, which report was based on a conference in December 1907, finished by the Committee's Chairman, Harvard Professor Edward V. Huntington (of apportionment fame) in 1911, and then published by the Society for the Promotion of Engineering Education in 1912. In it, the two most important topics of algebra are the transformation of algebraic expressions, and the solution of equations. Ratio and proportion, variation, inequalities, and progressions are also mentioned briefly. Many other topics which are part of traditional high school mathematics are summarized, and calculus is the culminating experience. My reason for bringing this up is that one of the members of this AMS committee (remember that this work preceded the founding of the MAA) was Charles P. Steinmetz, consulting engineer for the General Electric Company. It becomes clear in this report that Steinmetz was telling the rest of the committee that the study of complex functions was essential to electrical engineers. This caused a furor. I quote Professor J.B. Webb: "After Dr. Steinmetz, a layman, produced his book on the treatment of alternating currents, using complex variables, there was less objection made to them. Now I say it is a disgrace that it should be necessary for a layman to show professional teachers that a certain part of mathematics is needed." Professor G. H. Morse from the University of Nebraska states that he used complex quantities in teaching alternating currents, and that the same was happening at Illinois and at Purdue. One implication of looking at algebra from the point of view of the technical workforce is that we will need to exchange ideas with, and listen to, their employers.

That was about complex numbers in the university. What about complex numbers in high school? The current discussion of this issue illustrates how the point of view on the relation between mathematics education and applications of mathematics is evolving. The traditional difficulty with complex numbers is that between their introduction in second-year algebra for the solution of quadratic equations, and the proof by power series in second-year calculus that $\exp(ix) = \cos U + i \sin x$, almost no use is made of complex numbers. Traditional applications are numerous, but depend on the notion of frequency and on the formula I just mentioned. De Moivre's Theorem is often taught in trigonometry, but doesn't get used for applications at this point. The possible introduction into the curriculum of difference equations, or if you prefer, dynamical systems, can provide for the first time some possible applications for complex numbers. If the characteristic equation for a second-order linear system with constant coefficients has complex roots, then the solutions will exhibit oscillations, and their study depends only on De Moivre's Theorem, not on the notion of frequency! If we now wish to decide whether or not the study of complex numbers in high school should compete successfully for some of the limited available time, we might want to see some possible applications of such systems. Two come to mind, both mentioned in Jim Sandefur's latest book (1993): Paul Samuelson's model of investment in the economy, and a difference equation model for the oscillation of a spring. Should a decision on the teaching of complex numbers in high school depend on the appeal of these models? Who knows of others?

Modeling as a central and even organizing theme for high school mathematics has appeal from the point of view of the large number of students who prefer their mathematics to be useful. It has even greater appeal as a philosophical vehicle for keeping students together than might otherwise be possible. All the destinations of high school mathematics students, whether it be the work force directly after graduation, or 2-year college, or various 4-year college majors, have in common the need to *use* mathematics. The need and desire to learn how mathematics is used can unite students over all the destinations we have just mentioned. This, in my opinion, is part of the philosophy of the Pacesetter program of the College Board, which is now in its first year of experimental teaching. It is intended as a capstone high-school mathematics course.

Much more comprehensive is the 3-year high school curriculum called ARISE, which is being developed by COMAP under a grant from the National Science Foundation. Here all the mathematics for grades 9 through 11 will be derived from, and connected with, real situations the students examine and model. Four units which were first developed last summer and are now in the first throes of gentle field testing are on Codes; on Motion such as exemplified by a movie marquee, a hot-air balloon or, eventually, morphing; on the Landsat satellite and its usefulness; and on the Moose population in Adirondack Park. The mathematics studied includes, for example, linear functions, parametric representation of motion, similarity, and the distinctions between linear and exponential models. Graphing calculators for each student and computers for each classroom are assumed. Opportunities arise for lots of other goodies, such as some new feeling for the significance of "one-to-one onto" through codes, or the multiplication of signed numbers through parametric linear equations and their representation on the graphing calculator. I shall leave it to my friend Sol Garfunkel to correct these hasty opinions.

It is too early to tell how such a dynamic relationship of algebra and modeling will thrive, but it is certainly exciting to be a part of the experimental process. Here are some other key features of a curriculum in which the needs of employers are considered important:

Data analysis will play a critical role. Its inclusion in school mathematics was pioneered by USMES 25 years ago, and received big boosts from the Quantitative Literacy Project of NCTM and ASA, and from the Conference Board report on what is and is not still fundamental in school mathematics. More recently, the Contemporary Pre-Calculus book written at the North Carolina School for Science and Mathematics (1991) and published by Janson shows what a powerful motivation data analysis can be for all kinds of algebra developments.

The use of spreadsheets on the computer, and other forms of simulation, will also play a great role in making algebra more useful, more timely, and more exciting.

Some other aspects of algebra whose importance is greatly enhanced by the prospects of its usefulness on the job are symbol sense, stability, and "back-of-the-envelope" approximation. Symbol sense as an algebra analog of number sense in arithmetic was, as far as I am aware, first discussed by Jim Fey in his chapter of MSEB's "On the Shoulders of Giants." For example, the need to extract different information from an algebraic expression leads to alternate but equivalent forms being preferred—much as different modes of arriving at a numerical answer compete in arithmetic. A sense of the possible effect of a particular coefficient on the behavior of a function is in some cases much like computational estimation. A feeling, developed by experience, that expression B could not have been derived from expression A, and that therefore I must have made a mistake, may be more like measurement estimation. Symbol sense is likely to be a very practical outcome of algebra, and deserves much further study. The duality between form and content which Jim Kaput emphasizes can also be interpreted as part of symbol sense.

Stability is a very practical aspect of many mathematical models of the real world, and deserves to be included throughout the curriculum. A small error in measuring each side of a box will create how much error in its volume, in its surface area, or in the length of a ribbon around it? A calculator makes such really useful investigations actually possible. If you are going to triangulate, where should you stand? Why is it so difficult to hang a picture the way some other family member wants it hung? A talk at the 1994 NCTM Annual Meeting, and hopefully a paper, will go further into this fascinating topic.

Back-of-the-envelope computations, rough approximations on how something will come out, are bread-and-butter for users of mathematics in business and industry. Often you cannot investigate all possible alternatives in a given situation in detail. You must eliminate many of them by quick approximations, and concentrate on the most promising ones. Another delightful example of what the employer wants from algebra.

There has been discussion at this meeting of "big themes." Big themes in mathematics are often mentioned—function, chance, change, structure, number, operation, dimension, and space are just a few. Here is the beginning of a list of big themes of mathematical modeling: stability, optimization, absolute standards versus special cases, estimation, encoding, choosing among alternatives, the algorithmic point of view, linearity, feedback, system, signal and noise, and periodicity. I have intermixed form and content themes.

Having shown, I hope, some of my enthusiasm for modeling in the algebra (and mathematics) curriculum, let me express one particular worry which is unresolved in my mind. Every real situation is more complicated than you can possibly handle mathematically, and modeling involves critical decisions on what to keep and what to throw away. In all honesty, part of those decisions, for a professional, is prior knowledge of the mathematical area which is likely to result. You may be less likely to want to keep that cubic term if you know that it will probably make further analysis hopelessly difficult. Try it first without that term, and see if the results make sense in the real world. If students are learning their mathematics from the motivation of the model, how can they make such decisions based in part on knowing the mathematics already? I suppose that it will be a spiral process. Arthur Murray didn't advertise that "If you can dance, you can walk;" it was the other way around. I want to see the spiral work.

In this brief paper, we have taken a broad view of technical work force. The mathematical needs of everyone from apprentice technical assistants to R & D engineers have been in the back of our minds. In the current reality of mathematics education, the question of remedial algebra cannot be ignored. Developmental mathematics has all too often been nothing more than a repetition of what the student has failed to learn once before—only louder! In my opinion, this is less likely to work than a new approach. Both the applications point of view and the new technology are promising in this connection. An algebra experience centered around the usefulness of the subject may succeed where previous attempts at rote learning did not, especially with an older student. The use of graphing calculators in algebra has recently been shown to be really effective in improving both the spatial visualization and the conceptual understanding by girls at the college level (Shoaf-Grubbs 1992), and I would expect this approach to have broader applications. Incidentally, she also found that the calculator improved the connections that students made across representations—tables, graphs, and formulas.

I should like to close with reference to a point of view on science and mathematics education which I have recently read and found both intriguing and disturbing. For most of the past 300 years, it is claimed, the world has been creating science and mathematics, and producing people to do this creating, at an exponentially increasing rate. This can be documented, for example, by the growth in the number of scientific journals. Now if you say it this way, it is clear that this can't continue, and in fact the curve starts looking linear rather than exponential about 1950 (cf. Goodstein 1993). I think that government and industry fueled the exponential in employment for perhaps another 20 years after that. But our entire educational system in mathematics and science has been designed, and even optimized, to feed this greedy exponential. We probably produce 5 times the number of Ph.D.'s needed for the preservation of the species, and only the exponential growth in academic and industrial research positions kept this system going—until recently! We simply may not need that many any more!

But if this were to be true, then we no longer need to design our school and college educational experiences for the primary purpose of producing future Gausses (as Dan Teague likes to say). The algebraic experience in many high schools is optimized so that any potential future mathematician will be thrilled—and will not be overlooked. It isn't quite so good for everybody else! It seems likely that a much broader range of students will be thrilled by an algebraic experience centered on the potential usefulness of the subject. If it is true that enough future mathematicians will rise to the top regardless, then this may be a net gain for society. It was Bogoliubov, I think, who recommended that all mathematics Ph.D.'s should be in applied mathematics, and a few of these should then be selected to study pure mathematics! I don't wish to mislead the discussion with such a wild extrapolation. Let me just conclude by reminding us all of the fact that the usefulness of mathematics is extremely broad and all-encompassing, and that the needs of employment, when we look at them in detail, may provide an excellent focus for the rethinking of school mathematics in general, and algebra in particular.

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Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce

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A response to Algebra and the Technical Workforce

It is, of course, impossible (as usual) to find anything in Henry's talk to disagree with. To a large extent, his sentiments represent the philosophical underpinnings of all of COMAP's efforts over the past 13 years. Our work on the ARISE project is in many ways a natural consequence of these beliefs. We are attempting to create the curriculum materials that embody these ideas and are immediately usable by teachers and students as their basic high school offerings.

Henry has made the case for presenting algebra in real modeling situations more eloquently than I can. I will not attempt to restate his arguments. But the potential problems he raises need to be addressed. All of today's successful modelers (at all levels) came through the standard secondary school algebra courses we now wish to change. Henry's early algebra education and mine looked quite alike. Modeling and applications came to all of us as an epiphany. We learned to factor trinomials, we drilled and practiced symbol manipulation, but somehow by some mysterious process we developed an ability to apply these tools.

How did this noncontext driven curriculum give rise to (at least some) people who now see so much of mathematics in context. The truth is that we don't really know. Commonsense argues that a modeling-based introduction of algebra (and other mathematics) should give rise to more and better appliers of mathematics in all of Henry's senses. But we have little hard evidence.

I believe that the curriculum we are designing will get us closer to Henry's goals. Perhaps this is an article of faith. But we must find out; we must experiment. The potential good outweighs the potential problems.

Moreover, I firmly believe it is the tumult that counts. Curriculum reform implies teacher education both in-service and pre-service. Everything is in flux. What do the standards mean in practice? What is modeling? What do I need to know about election theory, coding, morphing? Am I teaching mathematics or political science or physics? Am I a lecturer or a classroom manager? Which graphing calculator should I use? Which computer? Do I let my students use them on the test?

I would argue that even though we have no assurance that our new curriculum will be better than our old, this amount of change and uncertainty will by itself produce positive results. Everything is on the table. We are re-examining our most fundamental beliefs and practices. Through all of this, we become a stronger community and our students the ultimate beneficiaries.

Finally, let me restate Henry's concluding remarks. Mathematics education should not be about the education of mathematicians, but rather the much broader segment of the student population who will need mathematics in work, as citizens, and in their daily lives. I firmly believe that the curricula which emerge from this round of reform will continue to produce outstanding research mathematicians. However, even if that is not the case, what we are attempting to do—namely, to provide a solid, useful mathematical experience for the widest possible student audience—is simply put, the right thing to do.

This position will not be universally accepted. Those of us who publicly express it may very well feel the wrath of our colleagues. But it is necessary to be direct about this message. Henry is fond of saying that all systems operate optimally, we simply need to determine what they are optimizing over. The reason that this debate is so important is that we are consciously attempting to change what the nation's system of mathematics education optimizes. Consensus on this point will ultimately be the most important contribution we can achieve.

A Cognitive Perspective on the Mathematical Preparation of Teachers: The Case of Algebra

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Recently, I presented a task to a class of senior mathematics majors aspiring to become secondary mathematics teachers. The class, which I taught, is the capstone course in the major—a course on mathematics curriculum and instruction in the secondary school. The task was as follows:

> Describe what you know about each of the following math ideas: slope, covariation, rate of change, tangent, derivative. Which, if any, of these ideas do you perceive as related? Explain.

I assigned the task after we had worked on various problems involving rate of change—constant, variable, and average rate of change. It was in the context of solving such problems that I became aware of the students' instrumental understandings of what I consider to be key ideas of the secondary school mathematics curriculum. Curious about what relationships they saw among these ideas, I decided to probe the nature and depth of their understandings.

I regarded the task as important for several reasons. The ideas of slope, covariation, rate of change, and derivative are all closely tied to the notion of function, and the notion of function is thematic to the algebra curriculum recommended in current reform documents (National Council of Teachers of Mathematics 1989; *California Mathematics Framework* 1992), not just for the middle and high school grades but also for kindergarten through fourth grade. Another reason for giving the task to my students was to chip away at their compartmentalized image of the mathematics curriculum. I sensed that, for most of them, slope was a topic in coordinate geometry which also popped up in calculus; tangent was a topic in geometry as well as in trigonometry; derivative was a topic in calculus; and rate of change was a topic that did not have a specific compartment in the curriculum, but which appeared sporadically throughout, as in calculus word problems. I was unsure about what the students would say about covariation. I was hoping that the assignment would cause them to reflect on what they knew about these ideas and that from such reflections they would emerge with some grasp of the conceptual links among them.

The students took the assignment seriously, writing lengthy essays for each part. What follows is a summary of their responses:

- Slope was generally described as the difference of the ordinates of two points divided by the difference of the abscissas. Some students included graphs to illustrate the elements in the computational definition they gave. Many students indicated that slope was a measure of the steepness of a line and stopped there. A few students referred to slope as the ratio of vertical change to horizontal change and explained that in this sense slope was like a rate of change.
- Tangent was identified by most students in the geometric sense as the link between slope and derivative. A few students related the trigonometric tangent to the slope of a line, offering the tangent of the angle formed by the line and the x-axis as an alternative method for computing the slope of the line.
- All students gave the textbook definition of the derivative of a function. Many indicated that derivative, slope, and tangent were related since "the derivative is the slope of the tangent line," and included graphic illustrations. Few mentioned that the derivative itself was a function. No one commented on the similarity between the computational definitions of derivative and slope.
- Most students indicated that covariation was not a recognizable term in the mathematics curriculum and made no attempt at describing any meaning for it. One student who looked up the term came up with a definition for "the covariance of two random variables"—the statistical cousin of the term she sought, but did not identify the blood relation. Only one student identified covariation as the fundamental link among the five ideas. He was an engineer returning to the university for a teaching credential in mathematics.
- Rate of change was connected to derivative only because derivative is instant rate of change. Only a few students explicitly mentioned what it was that changed.

In summary, it is fair to say that except for covariation most students were familiar with the terms and their definitions, and were able to identify and recall many of the calculational techniques associated with the concepts. However, except in the case of a few students, there was little in their responses to suggest that these ideas were held in relation to one another; that is, that they were part of a conceptual system. In several cases, incoherent responses were telling of a lack of clarity in the students' understandings of the concepts. To illustrate, consider the following excerpt from one of the papers:

> Slope is a measurement of the vertical change with respect to the horizontal change of a line. This measurement of two distances (horizontal and vertical for slope) is the notion of covariation; two things varying with respect to each other. The rate of change is a measurement of the slope of a line. With the tangent, it is my understanding (outside of triangles) that the tangent is merely the slope of a curve (line) at a certain point. Likewise with the derivative; it can also be seen as the slope of a curve (at an interval). Connecting the tangent and the derivative, the tangent is the derivative of a function at the evaluated interval.

I chose to start by relating this experience with my class because it illustrates concretely the importance of addressing not only *what* teachers should know but also *how* they should know what they know. This is hardly a profound or insightful statement, but one that needs to be stressed and clarified. In this paper, I address in a global manner both what mathematics teachers should know and how they should know it in order to teach algebra in grades K-12.

Knowledge Base for Teachers

For discussions about the mathematical preparation of teachers to be productive, it is necessary to clarify what it is that they should be prepared for. Moreover, to be able to specify what sorts of experiences they should be able to provide their students, we need to be clear about what we want students to learn. Thus, discussions about teacher education can be productive only to the extent that they are informed by a consensus and a clear image of what students at different levels of schooling should learn. Furthermore, our discussion needs to be informed by the rapidly growing research literature on the learning and teaching of algebra, and by the recommendations set forth in reform documents regarding the development of understandings and competencies relevant to learning algebra.

A K-12 Algebra Curriculum: A Focus on Functions

For several years a primary concern in mathematics education has been how to incorporate aspects of algebraic reasoning into the elementary curriculum (Wagner & Kieran 1989). That is to say, students' study of arithmetic is supposed to prepare them for algebra, but for the majority of students it does not do so. How transitions between the study of arithmetic and the study of algebra might be made is an important area of research in mathematics education (see Kieran 1989 for an excellent review). Difficulties caused by that transition is another important research area (Booth 1981; Herscovics & Kieran 1980; Vergnaud 1988).

Our research (Pat Thompson's and mine) over the past 6 years started with the idea that competence in algebra is founded in competent quantitative reasoning.[1] Since then our work has suggested that an emphasis on quantities and quantitative reasoning early on in the students' study of arithmetic provides a foundation that is not only necessary for the study of algebra but that can facilitate students' access to it.

Understandings of quantity and quantitative relationships in dynamic contexts are foundational to the functional approach to algebra advocated in curriculum reform documents such as the NCTM *Standards* and the *California Mathematics Framework*: an approach that focuses on functions, functional relationships, variables (in the physical sense of varying quantities), and other function-related ideas. In *A Call for Change*, the MAA endorses such a functional approach by specifying related competencies for the preparation of K-4, 5-8, and 9-12 teachers.

Curricular reform documents, however, do not explicitly address the central role of quantities and quantitative reasoning in a functional approach to algebra. By this omission, they fail to provide proper guidance to teachers on how they might approach a treatment of functions in the early grades. From our perspective, to ground the development of algebraic thinking on the notion of functions and functional relationships without, in turn, grounding these on understandings of quantities and quantitative reasoning in dynamic situations, is like building a house starting with the second floor. The house will not stand.

While the desirability of a functional approach over other approaches to bringing algebra into the early and middle grades may be debatable, I will not undertake such a debate here. Rather I will take this forming consensus as a point of departure and try to clarify the understandings and competencies teachers will need in order to implement the recommended instruction in those grade levels. I will also mention some of the difficulties they are likely to encounter. In regard to high school, I will discuss issues of generality and structure, suggesting ways in which a perspective of algebra as generalized arithmetic might be brought to complement the developing functional perspective introduced in the earlier grades. Issues of representation (notation and conventions) and the role of abstraction in the study of structure will be addressed along with their implications for the preparation of secondary teachers.

Implications of a Functional Approach for the Preparation of Teachers

Because we view quantitative reasoning as foundational to the study of algebra, we believe that the preparation of all teachers should be aimed at bringing about a reconceptualization of arithmetic in terms of quantity and quantitative operations. I say all and not just elementary and middle school teachers because such a reconceptualization provides a foundation for understanding functions that few secondary mathematics teachers appear to have. The case of my students, related in the beginning of this paper, is a case in point. Few of them viewed the concepts in the task as related to the idea of functions, except for derivative, and only one recognized the terms as having to do with covariation of two or more quantities. In their responses, few students used a language that spoke of an awareness of the concept of functional relationships and covariation as the common conceptual thread among the ideas in the task—even after probing. For example, in response to the students saying that slope was synonymous with steepness, I presented several graphs of $y = 0.5x - 1$ in separate sets of axes with different scales. I then asked them to compute the slope of the line in each case and posed the question: "If slope and steepness are the same, then how come these lines have the same slope, but are not equally steep?" Although the students recognized the effect of the scale on the steepness of the lines, they were unable to offer an explanation of what slope actually measures or of its relevance as a mathematical idea.

To explain what a reconceptualization of arithmetic in terms of quantity and quantitative relationships might entail, I first need to clarify what we mean by quantitative reasoning and by several other constructs that are essential to such a conceptual restructuring.

An Example of Quantitative Reasoning

Here is an example of what we mean by quantitative reasoning. Two approaches to solving the following problem are depicted: an algebraic approach and what we call a quantitative approach.

> *Brother and Me*
>
> I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetskii 1976, p. 160)

An algebraic approach would be to set up an equation, such as $(6+t)d/40 = t\,d/30$, and then solve for t. This is a standard approach, and any problem like this one typically appears in an algebra textbook.

On the other hand, this problem can be approached in a much more "situation-sensitive" manner. Here is an example of one such chain of reasoning:

- Imagine myself and brother walking: I see brother in front of me, and all I notice is the distance between us. What matters is how long it takes for that distance to become zero.
- The distance between us shrinks at a rate that is the difference of our walking speeds.
- I take 3/4 as long as brother to walk a given distance, so I walk 4/3 as fast as brother.
- Since I walk 4/3 as fast as brother, the difference of our speeds is 1/3 of brother's speed.
- The distance between us shrinks at a rate that is 1/3 of brother's speed. So the time required for that distance to become zero is 3 times the amount of time brother took to walk it.
- Since brother had a headstart of 6 minutes, I will overtake brother in 18 minutes.

If we take this example as illustrative of a type of reasoning—what we call quantitative reasoning—and if we take quantitative reasoning as an objective of arithmetic instruction, then problems like the one discussed above can be included in the middle school and junior high curriculum.[2] To take quantitative reasoning as an objective of instruction, however, it helps to have a clear, detailed image of the mental operations and conceptual structures that enable quantitative reasoning to happen. A complete discussion of such a model can be found in a paper by P.W. Thompson (1990). Here I will simply describe some key constructs.

But before continuing, a caveat is in order. "Students' quantitative reasoning" is almost an oxymoron. For the most part, students do not reason quantitatively in school mathematics. Textbooks and curricula do not promote quantitative reasoning (Fuson, Stigler, & Bartsch 1989; Porter 1989; Stigler, Fuson, Ham, & Kim 1986). Many sources suggest that students rarely reason in terms of quantitative operations. Rather, their thinking is dominated by numerical operations, which are unrelated to quantity (see P. Thompson 1990). It would be surprising to find many teachers teaching for quantitative reasoning since a large portion do not reason quantitatively themselves (Post, Harel, Behr, & Lesh 1991). So, from the beginning we are in the somewhat awkward position of speaking of a particular type of reasoning that is rare, but which should be a primary aim of instruction in the elementary, middle, and junior high grades—particularly of instruction dealing with functions and functional relationships. Our work has shown that it is difficult for students to develop a grasp of change, variability, and covariation—all key to understanding functions—when their thinking is confined to reasoning with numbers (as opposed to quantities) and numerical operations (as opposed to quantitative operations), even when that thinking is grounded in contextual settings.

Constructs Relevant to Quantity-Based Reasoning

Several key constructs form the core of the conceptual structures that enable quantitative reasoning. An explanation of these constructs is necessary in order to speak with precision about cognitive objectives for which instruction might aim. A brief description of those constructs follows.[3]

Quantity: A quality of something that can be measured—for example, my height, the area of this room, barometric pressure, population density, someone's age.

Basic Quantities: Number of things, difference, ratio, and rate.

Difference: A difference of two quantities is the quantity by which one of them exceeds or falls short of the other.

Ratio: A multiplicative comparison between two quantities.

Rate: A multiplicative comparison between two quantities where one quantity's value is conceived as varying in constant ratio with variations in the value of the other. (See Thompson & Thompson 1992, for a more elaborate and precise treatment of rate.)

Quantification: A process of direct or indirect measurement.

Quantitative Operation: A quantitative operation is conceived when two quantities are taken to produce a new quantity.

Reconceptualizing Arithmetic: A Quantitative Approach

To take quantitative reasoning as an objective of instruction in arithmetic, teachers will have to reconceptualize the arithmetic curriculum in terms of quantities and quantitative operations. This will entail the development of a conception of arithmetic instruction that is grounded in situations in which students have to reason in terms of quantities and quantitative relationships, with the objective of using notation (arithmetic) to represent their reasoning. This approach offers students opportunities to "reason algebraically" in arithmetic. By reasoning algebraically in arithmetic we mean

- Reasoning about relationships;
- Using arithmetic representationally;
- Reasoning about quantitative situations with diminished attention to numerical information—numerical information is understood as being "incidental;"
- Reasoning about quantitative situations in the absence of numerical information; and
- Constructing formulas.

In short, to reason algebraically in arithmetic means to reason relationally about quantities in some contextual situation. It is important to note that to us the phrase "reasoning algebraically in arithmetic" does not necessarily entail doing calculations, reasoning with numbers, or manipulating symbolic expressions.

Quantity-Based Algebra

Students do not have to undergo the radical shift in their thinking that traditionally has marked the transition from arithmetic to algebra. The difference between quantity-based arithmetic and quantity-based algebra is one of representation. While in arithmetic we use the notation of arithmetic as the language of representation, in algebra we must introduce a new notation. Now we deal with situations in which the value of one or more quantities vary, therefore we must use symbols other than numerals to represent the quantities' varying values. Thus, the transition is marked by the introduction of letters as variables in their literal sense—to represent the varying values of a specified quantity in some context. The key idea, the idea of representational uses of notational schemes, however, is not a new one. I should clarify that by stating that the transition can be a smooth one I do not mean to imply that students will acquire instant facility in the use of algebra notation.

An example can help illustrate how representational uses of algebra notation can be introduced in quantity-based algebra:

Banana Peeling Contest

> Ivonne and Daryl are professional banana peelers. Ivonne peels an average of 5.5 bananas per minute; Daryl peels an average of 3.4 bananas per minute.
>
> Ivonne decided that she would get a head start (she cheats), so she peeled 15 bananas at home and carried them to Ivonne and Daryl's Banana-peeling Office. When she walked into the office she found that Daryl had gotten there early (he cheats, too) and that he had peeled 93 bananas. She began immediately to catch up.
>
> In this situation, let x stand for the number of minutes that have gone by since Ivonne began to peel bananas at the office.

a. Do this for each of the following formulas: Fill in the blank with a name for the quantity that the formula evaluates.
____ ___ = $5.5x$
____ ___ = $3.4x$

b. Do this for each of the following quantities: Write a formula that evaluates the named quantity:

Total number of peeled bananas Ivonne has x minutes after arriving at the office ____

Total number of peeled bananas Daryl has x minutes after Ivonne arrived at the office ____

c. What do these open sentences stand for in terms of the situation in which Ivonne and Daryl find themselves:

$5.5x = 3.4x + 93$
$3.4x + 93 = 5.5x + 15$
$3.4x = 5.5x + 15$

d. In how many minutes after she starts peeling bananas at the office will Ivonne catch up with Daryl?

The task illustrates the use of algebraic notation as a representational language. Algebraic expressions are introduced in contextual situations as formulas that evaluate embedded quantities. (We used part d of this task to have students represent values with expressions, graph functions, interpret the graphs in relation to expressions and functions, and interpret points of intersection in relation to functions and quantities.) In tasks similar to the one shown, equations are introduced as (1) statements of the equivalence of two expressions or formulas for a single quantity's value, or (2) as a formula for a quantity's value together with a value that the formula must yield. The latter affords the conceptual connection between the notion of an algebraic variable as representing the varying value of a quantity and the notion of a variable as an unknown (fixed) value of a quantity. The metaphor of a "freeze frame" in a film conveys the idea of the variable as an unknown.

In quantity-based algebra the fundamentals of the "grammar" of algebra are developed in a way that parallels current thinking on how students should learn the rules of regular grammar: purposefully, to clarify written communication of ideas. It echoes the statement in *Everybody Counts* (1989): "Doing mathematics is much like writing. In each, the final product must express good ideas clearly and correctly, but the ideas must be present before the expression can take form..." (p. 44)

Two consequences of teaching arithmetic and algebra with a quantitative reasoning emphasis are that (1) students are more likely to develop an appreciation of algebra as a powerful tool for supporting complex reasoning (recall *Brother and Me* and the complexity of a quantitative approach to solving it); and (2) we may have to rethink what problems to consider as algebra problems—what appear as algebra problems in many textbooks and tests may turn out to be complex arithmetic problems.

Here is an example of a problem that makes a compelling case for the use of algebra and illustrates its power as a tool in reasoning.

> *An American in Paris*
>
> José is an American who lives in Paris. His annual income of $65,000 is subject to taxation by both the U.S. and the French governments. His tax obligation to the U.S. is 30 percent of what is left of his income after paying French tax. His tax obligation with the French government is 18 percent of what is left of his income after paying U.S. tax. What percent of his income is left after paying taxes? How much does he pay in taxes to each government?

Note that in this problem operational reasoning will not carry us very far because of the self-referential nature of the quantities. Contrast *An American in Paris* with the following problem which was submitted for inclusion in a standardized exam with the intent of testing students' skill in setting up equations to solve problems.

> Mindy's restaurant can seat a maximum of 102 persons. Thirty percent of the tables can seat at most 2 people and the rest can seat at most 4 people. How many of each type of table are there?

Students who are competent quantitative reasoners can solve this problem as follows:

- The ratio of tables-for-2 to tables-for-4 is 3 to 7.
- Since there can only be a whole number of tables, the number of tables- for-2 must be a multiple of 3 and the number of tables-for-4 must be a multiple of 7.
- 3 tables-for-2 and 7 tables-for-4 yield a total of 34 seats; 6 and 14 yield a total of 68 seats; 9 and 21 yield 102 seats. Done!

While there is nothing wrong with this student's reasoning, this solution demonstrates that a correct solution to the problem cannot be taken as a measure of the student's ability in the skill being tested.

Quantity-Based Instruction and Access to Algebra: The Bridge Between Out-of-School Mathematics and School Mathematics

The following excerpt was taken from an early draft of *The California Mathematics Framework* (1990). I include it for its relevance to teaching algebra.

> Students come to school with a wealth of experiences; they could be said to be experts about their own lives and the culture in which they are growing up. This

> personal and cultural expertise is a critical access point. Students need learning experiences that draw on their expertise and connect it with the experiences of others. Connecting the way things make sense outside of school and the way things make sense at school is especially important for students whose out-of-school culture is at some distance from the in-school culture. If students don't have ample opportunities to make connections between their personal lives and classroom learning, the sense-making skills that they bring with them will be less likely to be connected with classroom activities and their access to the curriculum diminished. In other words, explicit connections between out-of-school and in-school experiences increase the likelihood that students will connect their out-of-school and in-school thinking. Rich, complex activities in the classroom allow for several points of connection; simplified, abstract activities preempt the opportunity for personal connections and thereby diminish access.

Our work suggests that quantity-based arithmetic and algebra facilitates students making sense of the symbols and notational schemes of arithmetic and algebra. The language of quantities, as opposed to the language of numbers and symbols, provides a conceptual bridge between their out-of-school and in-school experiences.

Difficulties Teaching Quantity-Based Algebra and Arithmetic

We have learned from our work with teachers that this type of conceptual approach to arithmetic and algebra poses a great challenge for them, especially for those who are well versed in the use of arithmetic/algebra as a tool for reasoning in contextual problems. These teachers show a tendency to use a computational language which often gets in the way of their attempts to use everyday language to discuss mathematical ideas. This difficulty appears to stem from the way they have come to represent these ideas to themselves. They have come to use arithmetic/algebra representationally—they can read a situation into arithmetic or algebraic expressions, and they can use those expressions to represent their understanding of a situation. In other words, they are "good" quantitative reasoners themselves; they have come to use arithmetic/algebra in two ways simultaneously—as a representational system and as a formula system to express an evaluation.

Paradoxically, this very competence poses a shortcoming in regard to teaching. These teachers' quantitative conceptualizations are often encapsulated in the language of symbols, operations, and procedures. Thus, they have no other means outside the language of mathematical symbolism and operations to express their conceptualizations. The language of arithmetic/algebra serves them well as a personal representational system, or as a system for communicating with other competent quantitative reasoners. Yet, that language serves them poorly when trying to communicate with their students who know the tokens of the teacher's language, but have not constructed the meanings and images that the teacher has constructed to go along with them.

There are two consequences of teachers having rich quantitative meanings encapsulated within the language of arithmetic/algebra which often manifest themselves in teaching. One is that they often short-circuit what starts out as a rich, conceptual conversation with students by suddenly shifting to express a complex idea in the language of calculations. We have witnessed how students' eyes glaze over when this happens in the classroom. The second consequence is that these teachers often misinterpret students when the latter express themselves calculationally. The teachers either read too much into a student's statement (injecting their own rich meanings into the students' calculational statements), or they try to make quantitative sense of students' jumbled calculational statements when the students themselves have not attributed any quantitative meaning to their calculations.

Teachers' understandings of contextual situations in ways that they can make explicit only by means of symbols and procedures, and their imputation of those meanings to students' calculations and actions, are often the cause of breakdowns in communication between them and their students in the classroom. We have witnessed many such breakdowns. The rich quantitative meanings that the symbols and procedures embody for the teacher are often insufficient for understanding and adequately assessing the source of students' difficulty. In order for teachers to understand what holds their students back, they need a deep understanding of the subtleties and cognitive complexities inherent in understanding mathematics conceptually (Thompson & Thompson, in press a). In addition, in order to be able to help their students, teachers need another language besides the language of mathematical formalisms to express their ideas. They need an unaffected, nontechnical language (Thompson & Thompson, in press).

From Quantity-Based Algebra to the Study of Structures

Sfard (1992) notes that mathematics can be thought of as "a hierarchy in which what is conceived operationally, namely as a computational process, on one level is reified into an abstract object, or conceived structurally, on a higher level." She describes the historical evolution of algebra as "a constant attempt at turning computational procedures into mathematical objects, accompanied by a strenuous struggle for reification." Supporting the notion that ontogeny recapitulates phylogeny, Sfard draws a parallelism between the historical struggle for reification and the struggle of students making the transition from operational algebra to structural algebra. According to Sfard, the transition is problematic because it requires that "operational thinking be replaced by structural [thinking]." That is, it requires that students come to grips with the process/product duality of algebraic objects (e.g., they must be able to conceive 5*x*-square root 2 both as a computational process and as its product).

From the brief description of quantity-based arithmetic given earlier, one can see that attention to structure need not be postponed until the study of formal algebra. In quantity-based arithmetic and algebra, students deal with structural thinking at the same time they deal with operational thinking—both are viewed as essential aspects of using notation representationally and as such are treated simultaneously. The development of operational thinking need not precede the development of structural thinking. As discussed earlier, issues of generality of processes can be dealt with early on as students engage in analyses of contextual situations in terms of quantitative structures—that is, in terms of networks of quantities and quantitative relationships (Recall *Sally & John*). Expressions (numerical or algebraic) are used to describe both a solution process and a quantity's value. Thus, from the outset students are dealing with the notions of structure and generality, and switching between operational and structural ways of thinking. The fact that these experiences start early on in their study of arithmetic makes the transition smoother. What is currently problematic about the transition is that, in algebra, students encounter a way of thinking that is not only new to them but that clashes with the deeply ingrained way of thinking, namely operational, that has dominated their school mathematics experience.

From our experience working with teachers we have learned that a critical element in allowing structural thinking early on is the availability to both teachers and students of a natural and unaffected language—a language other than that of number and arithmetic operations which tends to force students and teachers into operational modes of thinking.

It is reasonable to expect that, because of its efficiency and power, once students have gained access to algebraic notation, that they will want to use it whenever appropriate. Yet, a number of studies (Clement 1982; Clement, Lochhead, & Soloway 1979; Harper 1987; Sfard 1987) have documented a resistance on the part of students to use algebraic notation and a tendency to use verbal methods instead. Sfard (in press) notes that students' spontaneous use of verbal methods—that is, when such methods have not been part of their instruction—speaks of the "inherent difficulty of the transition from an operational to a structural approach" which, she observes, parallels the history of the development of algebra.

I wish to offer an alternative explanation that is not quite as deterministic. The findings of students' preference for verbal methods may be more a reflection of how they have come to know the notation of algebra, than of an inherent difficulty in learning the language itself. The verbal, rather than the symbolic language, is the only sense-making language available to students who have not learned algebra notation representationally.

Preparation of Preservice and Inservice Teachers

To deal with the question of what should be involved in preparing teachers to provide students with quality algebra instruction, we cannot limit our discussions to preparing algebra teachers, because we cannot prepare "algebra teachers," we can only prepare mathematics teachers. Thus, in addition to clarifying what it is that they should be prepared for—as I have attempted to do—we must also address issues of beliefs, orientations, and imagery that cut across topics and courses.

Many, perhaps a majority of, prospective mathematics teachers have never experienced levels of conceptual understanding that current reform efforts intend for the students they will teach. To these future teachers, mathematics is about learning and performing rituals for getting answers to computational questions. Moreover, persons entering education programs commonly hold the very same beliefs and misconceptions about the nature of mathematics that research suggests are among the primary causes of U.S. school students' low mathematical competence. Thus, mathematics courses in the preparation of teachers, being based on the assumption that the courses will extend their (valid) mathematical knowledge, end up not addressing insidious beliefs and misconceptions held by these prospective teachers. Worse yet, they often end up reinforcing those beliefs and misconceptions which will dominate practical decisions these people will make about what is important to teach and what is important for students to learn.

Discussions about what should be included in programs for the preparation of mathematics teachers must extend beyond the specification of courses and course topics in the case of preservice teachers, and must include more than illustrative activities in the case of inservice teachers. Those in charge of the design of teacher education programs must deal with the issue of teachers developing appropriate images to guide their decisions and actions.

If curricular reform is to happen in classrooms, then teachers will have to teach from the basis of a conceptual curriculum. In order to be able to do this, they must develop a conceptual orientation toward the subject matter. Elsewhere (Thompson, A., Philipp, Thompson, P., & Boyd, in press), we have noted that a teacher with a conceptual orientation is one whose actions are driven by:

- An image of a system of ideas and ways of thinking that she intends the students to develop;
- An image of how these ideas and ways of thinking can develop;
- Ideas about features of materials, activities, and expositions, and about students' engagement with them that can orient students' attention in productive ways—ways that generate "methods" that generalize to other situations; and
- An expectation and insistence that students be intellectually engaged in tasks and activities.

Most prospective teachers go through teacher education programs without developing conceptual orientations. A case in point is that of my own students, described at the beginning of this paper. None of them had a well-developed system of ideas of which the notions of function, slope, covariation, rate of change, and derivative were a part, even after having completed 16 semester hours of calculus/analysis. Clearly, we need to look for solutions that do more than specify courses and topics prospective teachers should take.

In the case of inservice programs, we need to do more than prescribe activities aimed at learning specific pedagogical and managerial skills (e.g., how to use cooperative learning arrangements, portfolio assessments, graphing calculators, computer technology). We must make clear what the contribution of such activities to a coherent vision of mathematics learning and teaching is. We must ensure that they are assimilated into a coherent system of ideas, principles, and ways of thinking that can guide teachers' actions. Otherwise, their practical value is drastically diminished, for their implementation will be unprincipled and indiscriminating—just as in the case of students applying skills they have learned in a decontextualized and isolated fashion.

The "vision" of mathematics teaching sketched in the NCTM *Standards* is one many teachers have difficulty envisioning because it is incompatible with the images of mathematics teaching and learning they have abstracted from their experiences as students. By asking them to incorporate new activities into their teaching, when it is not clear to them (or us!) how these activities fit into a coherent system of ideas or how they help promote specific ways of thinking in their students, our instruction is flawed in a manner similar to the way we claim instruction in the schools is flawed. We must take care to insure that novel ideas, materials, and activities are integrated into mainstream teaching of mathematics in a principled, coherent manner, and that their contribution to students' learning is very clear to the teachers. Otherwise, the net effect of our efforts will be one of cosmetic change in the classroom, not of substantive change.

To prepare teachers to teach mathematics (algebra), we must help them build images of systems of ideas and ways of thinking that they will, in turn, want their students to develop. In addition, we must help them build images of how these ideas and ways of thinking can develop in their students. To make this happen may prove to be difficult for numerous reasons—at least two are important. First, it will require that we, ourselves, develop clear images of what we want our students to learn and ways of thinking we want them to develop. Furthermore, our actions will need to be driven by an understanding of how these ideas and ways of thinking can develop in our students. This will require that we develop a cognitive perspective in our own teaching—one that takes frequent notice of what our students are indeed learning from the experiences we provide. This will be difficult because such a perspective seldom underlies instruction at the college level. Second, students have to develop a commitment to creating these images and ways of thinking and work hard to do so. It should not be possible for them to get by with the minimum. This will be difficult because of unevenness in the level of expectation across courses and professors.

To be well prepared to teach, teachers must also develop clear ideas about features of materials, activities, and expositions that can orient their students' attention and thinking in productive ways. My experience has been that most prospective secondary mathematics teachers have very limited and impoverished images of the secondary school mathematics curriculum. How their college experience can help enrich those images is something we need to think about.

It should be clear that when specifying educational experiences for teachers, we need to go beyond specifying courses they must take and topics they should study. We need to focus on general principles of teaching and learning mathematics as well as on those that are specific to the teaching of algebra, be it elementary, intermediate, or advanced.

As we struggle with the issue of how to make mathematics accessible to more students, perhaps we also need to consider how to be more selective about who is inducted to be a mathematics teacher.

Those who can, do; those who understand, teach.

Lee Shulman

Notes

1. This idea is similar to a view of algebra as "generalized arithmetic," but is different in key aspects for we view arithmetic as also founded on quantitative reasoning.

2. Krutetskii (1976) considered this problem to be an arithmetic problem.

3. For a complete discussion of a model of the cognitive processes and conceptual structures that enable quantitative reasoning, which has provided the theoretical basis for our work, see *A Theoretical Model of Quantity-based Reasoning in Arithmetic and Algebra*, P.W. Thompson (1990).

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Preparing Teachers to Teach Algebra for All: Preliminary Musings and Questions

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In order to achieve "algebra for all students" teachers must believe that all students can and should learn algebra and must be prepared to develop and teach programs which include algebra for all students. They must have a clear picture of what algebra is and what its role in mathematics and in other fields is, and they must be prepared to teach algebra to students with tremendously varying backgrounds, needs, and interests. We at the college level are responsible for the preparation of these teachers. This responsibility encompasses the mathematics and education programs offered for prospective teachers, and the relationships of those programs and the faculty who design and conduct them. It also encompasses the preparation of elementary and middle, as well as high school, teachers. In considering the preparation of teachers, I found it useful to keep in mind both content knowledge, especially Harel's idea of concept image, and Buccino's content pedagogy, or pedagogical knowledge of content.

The mathematics community has long recommended that prospective high school teachers have a substantial mathematics background, usually equivalent to a major in mathematics. We might ask what aspects of the typical undergraduate mathematics program provides the appropriate background to help students become teachers prepared to teach algebra to all high school students. How are our courses designed to facilitate the development of those concept images? Do students in such courses gain a deeper appreciation for the roles of abstraction and symbolization in mathematics? Will they see a connection between the algebra taught in high school and these courses? What do we expect students to see beyond the definitions, theorems, proofs, and examples presented in the course? In what way are these courses designed for students to experience and reflect on the process of abstraction, of generalizing and building on systems encountered in other courses? Will students have opportunities to conjecture, explore, discuss, create, and explain, as well as to deal with formally stated theorems and proofs? Will they be led not only to understand, appreciate, and construct proofs but also to recognize and value these other important facets of doing mathematics?

We may wish to ask how these and other mathematics courses can be designed and taught to help students

- Develop a sense of algebra as a unifying language of (means of communicating in?) mathematics and perhaps to appreciate the history and development of algebra;
- Recognize the role algebra plays in connecting mathematics to areas outside of mathematics and learn how others use and view algebra;
- Develop an understanding of the high school mathematics curriculum, and the role of algebra in that curriculum, as it relates to collegiate mathematics and as it relates to the needs of all high school students, regardless of their future educational or career goals; and
- Encounter experiences in learning mathematics which will provide a rich background for learning to teach algebra to all students, including non-college bound students.

The focus in my thoughts for the preparation of secondary mathematics teachers is on the mathematics courses, since typically a program for prospective secondary mathematics teachers consists of 4 years of courses on mathematics and perhaps only one or two individual courses related to the teaching of mathematics. The experiences as students in learning mathematics thus plays a major role in determining secondary teachers' views of mathematics and beliefs about teaching mathematics in general and algebra in particular. It is essential, of course, that students also have exemplary courses and field experiences which further develop their pedagogical knowledge of algebra, an understanding of the diverse needs of students who will be in their courses, and skills in teaching to meet these needs. But these components of the teacher's preparation can only build on the firm foundation of attitudes, beliefs, and knowledge developed in mathematics courses.

Programs for prospective elementary teachers may include a year-long sequence of mathematics courses for prospective teachers, but more commonly include only one such mathematics course and at most one course on teaching mathematics. How will these prospective teachers develop an understanding of the need to prepare all elementary students for success in algebra and the ability to do so? How will they learn that mathematics taught at the earliest levels can provide the opportunities for exploring, generalizing, abstracting, justifying, and symbolizing in mathematics, which are the basis for learning algebra? How will they learn enough about the algebra which is to be taught to all students that they can design and teach a program which prepares their students well to learn algebra? How can the content and teaching of mathematics and education courses for elementary teachers instill in prospective teachers the concepts, attitudes, and skills to learn how to offer this kind of program? What is the responsibility of mathematics and education departments to offer and nurture such courses?

The preparation of middle school mathematics teachers is an area of particular concern to me, since these are the teachers who will provide many students, usually those deemed to be most capable in mathematics, with their first substantial experiences in algebra. It is very common that middle school teachers of mathematics have an elementary teaching certificate, and hence possibly only the mathematics described above. There may be no courses or field experiences specifically devoted to teaching mathematics to early adolescents. What should the preparation of these teachers be, and what is the responsibility for mathematics and education departments to offer such a program?

To this point the questions I have raised have been described in terms of preparing prospective teachers. Just as important is the preparation for teachers already in the field to help all students have success in learning algebra. Whose responsibility is it to help current elementary teachers—many of whom themselves fear mathematics and feel uncomfortable teaching it—prepare their students for success in mathematics at the secondary level? To help middle school teachers—who themselves may not know how to graph a linear equation—prepare to teach algebra? To help high school teachers—graph a linear equation—prepare to teach algebra? To help high school teachers—often with experience in teaching algebra only to college-bound students, with those courses designed as the first link in a chain leading to calculus—that all students can and should learn algebra? To design and teach courses in which all students can in fact learn algebra? I suggest that this is a responsibility of both mathematics and mathematics education faculty, and that we must grapple with how to meet these needs in tandem with the needs of prospective teachers.

And so we may wish to ask about the mathematics and mathematics education components of a program for teachers at any level: Where will the preservice or inservice teacher develop an understanding of the importance of algebra for all students, the belief that all students can and should learn algebra, and the knowledge and skills to offer such programs? How will faculty learn to use a variety of diverse teaching strategies effectively in reaching students with differing learning styles, backgrounds, and needs, thus modeling teaching attitudes and techniques for preservice and inservice teachers? If a mathematics program does not offer separate courses for prospective teachers, how should courses be adapted to meet the needs of these students as well as students with different career or educational goals? If an education program is too small to warrant separate courses on the teaching of mathematics, how will students interested in teaching mathematics gain the pedagogical knowledge of algebra and the skills to teach it? What avenues are there for offering courses which integrate knowledge and pedagogical knowledge of algebra? In programs for teachers or prospective teachers, how much focus should be placed on teaching algebra, as opposed to teaching arithmetic, geometry, probability, statistics, or topics from discrete mathematics to all students? How can mathematics faculty be encouraged to, assisted in, and rewarded for taking an active interest in prospective and inservice teachers of mathematics and programs for them?

I look forward to our discussion of these and many other interesting, important, and possibly difficult questions during our colloquium.

Algebra for All: Dumbing Down or Summing Up?

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Mathematical Sciences Education Board

My task—to sum up the manifold discussions of the last 3 days—is necessarily inchoate. Each of us takes from this conference a uniquely personal perspective based on the particular working groups and discussions in which we participated. I do not pretend that my remarks will provide either a synopsis or an executive summary. They are, rather, a personal perspective on what I have heard these last few days. I'll try to follow the conference mandate: to focus on a few big ideas, especially by providing a policy perspective, emphasizing open questions, disputes, and omissions.

My notes from the discussions cover an enormous variety of topics—far too much for a one-hour summary. So, I adopted two principles of selectivity:

- Omit ideas that have been widely discussed elsewhere during the last decade; and
- Omit issues that are not specific to algebra.

One consequence, for instance, is that I have made no attempt to cover the workshop's wide-ranging and enormously helpful discussions about teacher education—not because it is less important, but because it is part of a larger discussion that has been going on for quite a number of years and because it is not, except in rare instances, uniquely related to the subject of algebra.

These two filters reduced my notes by at least two orders of magnitude. What emerged was not much about AFA ("Algebra for All"), much less about the proposed new acronym of AFAFA ("All for Algebra for All"). Most of our discussions seemed to be about everything except algebra: teaching, literacy, modelling, mathematics, standards. The few drops of new ideas about algebra that these filters let through from the gush of words at this workshop suggest that a more apt acronym might be AA—not as some have suggested "Algebraists Anonymous," but more accurately, "Avoiding Algebra."

What is Algebra?

I'll begin with a brief review of various definitions of algebra, based primarily on the lectures by Victor Katz and Mike Artin:

- Calculating by rules (Early Islamic);
- Determining unknowns;
- Computations by signs and symbols (15th-16th C.);
- Employing letters in reasoning about numbers;
- Working with '*x*' (the typical student's definition);
- Study of arithmetic operations;
- Simple and general concepts, not cumbersome computations (20th C.);
- Study of general operations and operators; and
- The language of mathematics (and of commerce, ...).

Prior to Emmy Noether, algebra was what most students believe it still is today: calculating with letters to solve equations for unknowns. Only in the 20th century did the more general notions arise that gave algebra its great power—the power to become, in Mike Artin's words, the "language of mathematics." Of course, it is also the language of commerce and of technical discourse in many other areas of knowledge as well.

Before moving on I should remind you of a few of the lessons of history embedded in Victor Katz' very informative opening lecture:

- Historically, algebra was taught through artificial problems.
- Historically, algebra was taught to the few, not the many.
- Historically, algebra was taught not to find answers to useful problems, but to develop general problem solving skills.
- Historically, algebra of the 20th century evolved from—hence requires mastery of—the algebra of previous centuries.

These lessons lead to a key observation and a significant warning: "Algebra for All" taught through realistic problems with reduced emphasis on complex manipulation to provide skills needed for work violates all these lessons of history. Are we really ready for such a revolution?

Major Themes

The major theme of this conference is very simple: Algebra is broken but nonetheless essential. Algebra I—the current manifestation of "Algebra for All"—is not serving anyone well—neither those who fail nor those who pass. Fortunately, good examples show that we can do better: although the challenge of reform is daunting, we know the direction in which improvement lies.

Algebra is, in Bob Moses' apt phrase, a "new civil right." This idea encapsulates the best expression of the motivation for the whole conference—of the source of power that underlies the concept of algebra for all. It encompasses a host of powerful ideas:

- Algebra can democratize access to big ideas.
- Algebra provides evidence of the power of one's own mind—an unparalleled source of authority for youth.
- Tracking, the chief alternative to algebra for all, serves primarily as an engine of inequity.
- Effective instruction requires a respect for diversity grounded in multicultural perspectives.

A second major theme that I heard reflected in virtually all working groups is that algebra in school should serve broad purposes, not just job-specific skills. Its goal is not to convey the contents of a formulary, but to help students acquire the capacity to learn new things. Under this theme, I group a number of common conference understandings:

- Teach a few big ideas well, not many superficially.
- In order for algebra to serve broad purposes it is essential that we develop and value diverse ways through which students can succeed with algebra.
- Algebra should be viewed not as a particular course, as it most often is, but as a set of ideas, skills, and habits of thought. Indeed, algebra should be a strand everywhere throughout K-16, not just a separate course.
- Typical college courses fail to model good instruction, thus sending improper signals to students, schools, and prospective teachers.
- Examples should precede theory; the specific should come before the general.

Finally, throughout our conversations there ran a counterpoint of idealism versus reality. This led to a third area of consensus: In making recommendations, we must distinguish short-term (politically constrained) actions from long-term (unconstrained) vision.

The "Story Line" of Algebra

We all agree that algebra is in part about symbols. We also seem to agree that algebra is itself a symbol. There agreement ends. As I listened, I sensed that algebra is used by mathematicians, teachers, and the public as a symbol—sometimes as a codeword—for a number of different political and social agendas. Some examples:

- What parents studied in school—the embalmed memories of a bygone era;
- The contents of the algebra book—engraved authority;
- Job screening—a hurdle for quality employment;
- Requirement for graduation—a mark of rigorous education;
- Key to good jobs—the economic metaphor;
- Motivator for high expectations—algebra as the epitome of achievement;
- Symbol of failure and phobia—the worst memories of school mathematics; and
- Filter for success—a hurdle to ensure stability of the ruling class.

Yesterday Bob Moses asked for the "story line" of algebra. What can we say to inquiring skeptics to convince them that algebra really is important? Yesterday no one dared take up the challenge. We owe Bob—and ourselves—an answer. So let's provide it now. Write 2-3 sentences on why you believe algebra is so important. Pick one of two options according to your interests (see appendix A, Why Study Algebra?, for an edited compilation of the responses):

- Why algebra for all? (addressed to skeptical parents or school board members)
- Why take abstract algebra? (addressed to a college student or a prospective high school teacher)

Various speakers promised to tell us, at last, just what algebra is. None did. However, I found in my notes three categories of responses. Some thought in traditional terms such as:

Logic and proof
Use of variables
Solving equations
Deductive reasoning
Symbol Sense
Simple manipulations
Modeling

Others spoke in more abstract categories such as:

Generalized arithmetic
Study of relationships
Means to solve problems
Study of structure

Henry Pollak challenged us to think about new connections of algebra

To spreadsheets	To data analysis
To stability	To algorithms

These divergent perspectives may help explain why we have found it so difficult to define the essence of algebra, why we have so often avoided the very topic of this workshop. As the elephant to the blind man, algebra appears very different to its many different constituencies.

Language and Understanding

One of the more fascinating recurring themes of this workshop has been the various speculations about the relation of algebra to language. It began with Victor Katz's many historical examples of algebra problems posed and solved entirely in natural language. This strand continued with Mike Artin's ruminations about the relations of language, abstraction, numbers, and algebra.

Alba Thompson continued this theme by showing through many examples the consequences of disjuncture of words from symbols. Many students see no meaning in symbols, and their teachers frequently cannot convey meaning without using the very symbols that their students do not understand. For example: Why does $0 \bullet 3 = 0$? How can this be explained to a primary school student? Will the typical university course in abstract algebra help convey to the prospective teacher an explanation that a young student can understand?

Discourse about algebra is the key to keeping alive a capacity for reasoning and proof. Ten words that students understand may be worth a thousand meaningless symbols.

Algebra in the Workplace

A recurrent theme at this workshop is the changing role of mathematics in the workplace. Today's schools and curriculum are the legacy of an era in which shopkeeper arithmetic was the mainstay of workplace requirements for mathematics. Algebra for all is motivated in part by the assumption that algebra is the key to well-paying jobs (for individuals) and to a competitive work force (for society). Several themes emerged in these discussions:

- Most mathematics used in work is just high school-level mathematics. Very few individuals use college-level (calculus-based) mathematics in work situations.
- The curricular focus for work force issues is on the upper 2 years of high school—in Algebra II and its 12th grade successor. This is where the school-to-work and tech-prep themes are most noticeable. One must distinguish at this level three different intentions: pre-work, pre-college, and pre-calculus.
- Algebra is an "invisible culture" not recognized in the workplace. Neither employers nor employees are aware of the extent to which problem solving skills honed in algebra are used in everyday workplace situations. It takes an attentive observer to see the role played by mathematics in ordinary work
- Nevertheless, lots of good algebra can be found in the workplace, although little of it appears in the classroom. For many reasons—textbook tradition, teacher background, pedagogical difficulties—the "applications" of mathematics continue to be presented in isolation from the context in which they arise.
- Traditional curricula—particularly school algebra texts—teach mathematics first, then applications. The rationale is to first master the tools, then apply them to use. Modeling reverses that order: the problem context leads to the mathematics. A modeling approach, which better fits workplace needs, can clash with traditional curriculum since one cannot provide advance assurance of which mathematics will be covered or learned.

College and University

Whether because it is my natural interest or because it is just such a complex and important issue, the role of algebra in college seemed to command a lot of attention, especially in the working group sessions. I heard four broad persistent worries in these discussions:

1. The nature of high school-level algebra when taken in college—frequently by adults;
2. The need for sweeping reform in undergraduate mathematics in order to improve the preparation of teachers;

3. The "brick wall" students encounter in the first course in abstract algebra; and

4. The evolving demands for tighter coupling between college-level mathematical programs and the world of work, especially in the 2-year colleges.

First a bit of terminology. By high school algebra I mean to include all the algebra that leads up to calculus, including the contents of the course that traditionally has been called college algebra. This is, roughly, the algebra of the 18th and 19th centuries. By college-level algebra I mean to include linear algebra and abstract algebra—courses normally taken following calculus that form the heart of an undergraduate mathematics major. One of the major problems of algebra as it is practiced in today's schools is the lack of mathematical, pedagogical, and psychological connection between these two kinds of algebra—between the pre- and post-Noether views of the subject.

Several issues revolve around high school algebra:

- If high schools did their jobs well, three-quarters of all mathematics courses in community colleges would need to be totally redesigned. This is not an argument for maintaining the status quo—in which community colleges largely repeat the high school curriculum—but a reflection of the need to develop imaginative post-high school courses that meet the real needs of students for broader mathematics education.
- Colleges and universities do not give credit for remedial courses, so students take these courses under a financial and psychological handicap. What are the educational reasons for denying college credit to Algebra I while awarding it to French I?
- Most college faculty don't want to teach students who enter college still in need of high school algebra, nor do they give much of their leadership energy to making courses for such students effective.
- We need a new suite of post-Algebra II courses, especially in community colleges, that provide algebra-based materials in the service of major vocational programs (e.g., health, business).
- Placement testing in colleges and national exams taken by college-bound students impede innovation in high school algebra by establishing public

expectations for the traditional curriculum among students, parents, and teachers.

I am, reluctantly, focusing this summary on issues other than teacher preparation. I do want, however, to make one suggestion—actually, more like a challenge—about the need for rethinking the mathematics that is offered to prospective elementary school teachers. Suppose, as we hope for the future, that students enter college with a mathematical experience similar to that suggested by the NCTM *Standards*. We would then need a new course that might be thought of as "elementary mathematics from an advanced standpoint," or perhaps as "advanced mathematics from an elementary point of view." Such a course could serve well the mathematical needs of prospective elementary teachers; of future parents who need to understand mathematics in order to support their children's education; and of educated citizens who are seeking the benefits of a liberal arts education. These three constituencies share much in common—the need for deep understanding of elementary mathematics—as distinct from the professional who needs broad understanding of many parts of advanced mathematics. I suspect that a new course, perhaps incorporating themes similar to those in *On the Shoulders of Giants*, could serve all three purposes well.

The Brick Wall

Faculty in departments of mathematics have always known that a student's first encounter with abstract algebra—real mathematics—is often a traumatic experience. Abstract algebra and to a lesser degree the parallel course in elementary real analysis are often the first college courses that employ proof in an essential way. Students whose view of mathematics is shaped by 2 years of problems-oriented calculus enter these junior-level courses with a distorted view of the nature of mathematics. For many, they hit a brick wall and never recover.

Conferees who have had extensive experience with this phenomenon report that the situation is much worse now than it was, say, 20 years ago. Various explanations are offered. Students have changed—the population is more diverse, with more diverse aspirations and goals; calculus has changed—to emphasize skills and problems, not rigorous thinking; high schools have changed—to stress preparation for calculus rather than preparation for mathematical thinking. Probably all conjectures are valid: today's students who enter college-level algebra courses are expected to abstract from things they haven't experienced thus violating a common lesson of both history and pedagogy.

One conjecture emerged that intrigues me. I pass it on for your consideration, not as a definitive explanation, but as a way to view the problem that may harbor clues to possible solution.

The route from arithmetic to calculus is through high school algebra. During the past two decades there has been a clear shift in emphasis, reinforced in recent years by the introduction of graphing calculators, to emphasize functions (that is, pre-analysis) rather than operations (that is, pre-algebra). Does the functions approach to high school algebra impede development of key algebraic skills and intuition?

The following schematic diagram suggests two possible mathematical routes from grade school to college:

Grade	**Pre-analysis route**	**Pre-algebra route**
K-6	Arithmetic	Arithmetic Operations
5-8	Quantitative Measure	Algorithms & Advanced Operations
9-11	Functional Approach	Algebraic Approach
11-13	Elementary Functions	[???]
12-14	Calculus	Linear Algebra
14-16	Analysis	Abstract Algebra

The point of this chart is to suggest that the natural bridge to calculus provided by the pre-calculus emphasis in the upper years of high school does not exist for abstract algebra. Much of high school algebra now serves to introduce students to the function zoo so that they enter calculus with a rich base of examples on which to build abstract ideas. Where in the curriculum do students encounter a similar base of algebra-rich examples from which to build the abstractions required for modern algebra?

Big Ideas

Everyone seems to agree that algebra should focus on a few big ideas rather than lots of isolated skills. Part of the "avoiding algebra" subtext of this workshop has been the powerful reluctance of everyone to avoid proposing these key ideas. The explanation for this reluctance can hardly be an attack of modesty, since opinions have flown freely on many subjects on which we are in many ways less expert. It is hard to believe that no one has any idea about the key ideas in algebra.

Maybe the problem is that no one wants to go first. Well, now that it is less than an hour to go before the close of the workshop, this is our last chance to address the issue. I'll stop for five minutes to let each table discuss this question: What are the few key ideas of algebra that must be addressed in any curriculum? Each table will make a list—keep it short, please—and we'll see afterwards if there is any consensus (see appendix B for the list of key ideas produced by the conferees through this exercise).

Debates and Open Questions

Throughout the workshop I picked up hints of debates, disagreements, and areas requiring further investigation. Here are a few examples, enough to suggest that there is plenty of work ahead for all of us.

- Does the study of algebra really develop general problem solving skills? Was its historical use to empower the elite based on its real utility in developing broad political problem-solving skills, or might the persistence of algebraic hurdles for leadership been merely a convenient social filter designed to select those who had the advantage of a good education?
- Nearly everyone at this meeting seems to support the assertion that the same "algebra for all" in grades K-11 will serve equally well college-bound and work-bound students. Although the pace and depth of learning will differ from student to student, the core conjecture of algebra for all is that the same algebra is best for all, at least up to age 16 or so. Question: Is this an article of faith, or a claim rooted in evidence? We definitely need to learn more about how mathematical thinking is used on the job to determine whether it is defensible to adhere to a common core up until the 11th grade. (Of course, earlier fragmentation would reintroduce tracking, with all its negative social consequences.)
- "We need to understand better the nature of student understanding of algebraic concepts." This is what I might call my Alan Schoenfeld memorial sentence, since he wisely asserted it with increasing frequency as discussions seemed to drift into areas in which our ignorance exceeds our knowledge. I'd like to propose an iteration: "We need to understand better the nature of faculty understanding of student understanding of algebraic concepts." (I'm sure that John Conway can find a way to provide several more iterations.)

- Is "watering down" a code word for preserving the filter role of algebra? Is the public pressure for excellence just a subterfuge for gaining an advantage—excellence for the elite, mediocrity for the masses? There is some evidence that parental pressure for maintaining traditional courses validated by traditional exams is more a means of maintaining a competition which they believe their children can win than it is a true belief in the educational wisdom of these exams. Similarly, some university faculty insist on traditional entrance expectations not so much because they have concluded based on a review of all the evidence that these expectations yield better education, but because they fit better the courses the faculty are accustomed to teaching. There is a real danger that any kind of "new" algebra will be rejected either by the public or by mathematicians simply because it changes the rules of the game.
- Is there a better approach to elementary algebra for adults than repeating prior failure? Must everyone repeat the traditional syllabus, regardless of age, background, and educational objectives? Algebra is no longer just a subject taught to adolescents; more adults study high school algebra than study calculus and statistics combined. We need to find better ways to help everyone learn algebra.
- Related to the previous question is one of a narrower but nonetheless fundamental character: Can one delay symbol manipulations (in Algebra I) and then catch up later? If symbol manipulation is now less important for real applications, can it be delayed without serious harm to students' ability to learn subsequent algebra and abstract mathematics? Is there an appropriate time for learning algebra—as there is for language—that, once passed, is difficult to recover?
- Some people have been discussing a political and rhetorical issue: Do we want to stress "algebra for all" or quantitative literacy ("numeracy")? The former may have the flavor of medicine, the latter of a civic duty. What are the political implications? Voices were heard on all sides of this issue.
- Finally, a question based on reflection from many side conversations: Is an emphasis on what I might gently term "revolutionary" ideas (e.g., eliminating all algebra courses in school; utopian use of technology) just a way to avoid the

real problems of today's schools and colleges? This perennial tension between idealism and pragmatism was well reflected by the workshop participants—and in our discussions. It is a problem that we all face everyday in our work, one that we must bear in mind as we seek public support for improved mathematics education.

Challenges

I close with several challenges that arose from many parts of this workshop:

- Can we create a rich curriculum for high school algebra that truly integrates work force and academic perspectives?
- Can technology help develop algebraic and reasoning—not merely geometric—skills? Can there be an "algebraist's sketchpad"?
- Can the clients of high school algebra be persuaded to forego any part of the traditional repertoire of symbol manipulation? Can students learn mathematics if manipulations are slighted?
- Can teachers possibly keep up with the proposed changes? If we believe the popular diagnosis about weak preparation of teachers for today's curriculum, how can we imagine that they could teach a curriculum based on some of the more imaginative ideas at this conference? Could the United States afford the required retraining?
- Can we persuade the public that new algebra is real algebra? How can we allay the suspicion that algebra without mind-numbing manipulation is not just another example of "dumbed-down" curriculum?
- Can mathematicians retain control of algebra in the face of the new proposals for national certification of standards and performance expectations (e.g., the proposed National Education Standards and Improvement Council [NESIC])? Will the symbolism of algebra as a public good shift control from professionals to politicians in setting national standards? Who, ultimately, will decide what algebra is?

Gleanings and Grazings

Hidden here and there in our conversations and speeches are a variety of slogans that I leave as my parting contribution:

- Algebraists Anonymous
- Algebra as a new civil right
- A "Declaration of Ignorance"
- More math means more money
- Those who can, do; those who understand, teach
- Number sense, symbol sense, function sense
- 6 R's of college: remedial reading, remedial writing, remedial 'rithmetic
- Quantitative reasoning is algebra without symbols, algebra without knowing it
- "Algebra for all" is like a slow motion train wreck: if it fails, it will confirm the public belief that most students can't learn algebra

Appendix A: Why Study Algebra?

Conferees' Perspectives

Why require algebra? (addressed to parents and school boards)

- The information age is characterized by the generation of massive quantities of data; indeed, information is often identified as "the new capital." Increasingly, the ability to deal with large bodies of data is critical both to employability and to the exercise of the responsibilities and rights of citizenship. In order to do this, it is essential to consider variables within the data, and to view the data by imposing (or finding) structures on it. Algebra is the science of variables and models; knowledge of algebra is therefore the key to both employment and citizenship.
- Algebra helps develop students' ability to conceptualize ideas from the specific to the general. It helps them discover patterns among items in a set. Finally, it enables them to think through the various aspects of a problem-situation, to identify known and unknown quantities and the relationships among them, and to develop a strategy for determining the values of unknown quantities.
- Mathematics in all its facets provides ways of organizing, making sense of, and reasoning about patterns in the world of our experience. Many of the most significant patterns involve relations among variable quantities. Algebra is the basic set of ideas and techniques for describing and reasoning about relations among quantitative variables.
- Algebra is a powerful way to organize quantitative relationships which in turn can reveal new information (solutions). With an algebraic underpinning, one can see fundamental similarities in structure among diverse contexts.
- Algebra provides students with the knowledge and ability to manipulate situations (symbols) so that they may better understand what is going on around them in the world.
- Algebra provides a language and way of thinking which allows us to think about and communicate to others general properties and patterns of situations involving uncertainty, change, quantities, size, and spaces.

- Students need the general mental discipline and specific problem solving skills of algebra (e.g., manipulating symbols, solving for the unknown in order to compete for and hold an increasing number of technologically-related jobs in our economy).
- Ours is a technological society and becoming increasingly so. Algebraic thinking, the experience of looking quantitatively at situations and data (not the use per se of variables) is necessary to be a productive, contributing citizen, both through employment and consumerism.
- Algebra is the entry point for formalism and abstraction. It is required in order to understand the world around us and to function effectively in it as a citizen and worker. The nature of work and of jobs more and more requires the capacity for formalism and abstraction to formulate issues and to solve problems.
- Algebra is the one place in the high school curriculum where students are taught how to reason quantitatively and objectively about situations in the real world where the quantification cannot just be obtained by common sense and guessing. Elements include symbolic representation, precise computation, and interpretation of the symbolic (or numeric) variables in the context of the problem—in short, (mathematical) modeling.
- Algebra is becoming a universal language which describes relationships—covariation between two or more traits in many disciplines, businesses, and discussions for citizenship.
- Most of contemporary society, our technology, and our work depends upon mathematical models. Those models require an understanding of algebra. The ability to use algebra in constructing these models is essential to success in today's world.
- We need an algebra program that everyone should have. In it, students would learn to understand events and systems that are important in their work and in society more analytically and precisely in terms of their quantitative aspects; they would also learn to recognize and formulate key questions and learn to formulate arguments more persuasively.

- Algebra is a language one uses to represent the relationships of one set of variables to another set. The representation process involves tabular, graphical and symbolic symbols and rules. It is the language of mathematics, science, and business; therefore, all should learn to use it.
- By being successful in this "new" algebra, a student will explore relationships among quantities and ways of thinking mathematically which describe life in our complex world and the workplace.
- Algebraic thinking enables one to figure things out by thinking in terms of what's known, what's not known, and what are the relationships among them.
- Algebra is the "new literacy." Seventy-five percent of jobs require Algebra I and some geometry; therefore, to be prepared for society (and jobs), students need algebra.

Why study algebra? (addressed to high school students)

- Properly learned, algebra embodies both an analytic perspective—you learn to make sense of situations, to find out "what makes things tick"—and a symbolic language that helps to represent such situations and make sense of them. It's an important aspect of "mathematical power."
- Algebra at the high school level has two components. One might be called ideas (or concepts), and the other means of representation. Concepts include, for example, the distinction between linear and nonlinear phenomena, the idea of rate of change and the rate of change of the rate of change. Means of representation are used both to communicate with others, and to think about things ourselves. They include graphs, tables, finite differences, algorithms, recursion rules, and expressions like y = e^(-t) sin(alpha)t.
- We are going to learn to analyze situations, that is, to understand what is going on, and look for the principles of what is going on, and therefore what should be done. We will pull out ways of thinking that can be used elsewhere.
- Algebra introduces ways of modeling (describing) real-world situations. It involves use of symbols and graphs, and techniques for solving problems and making inferences. For example, $y = 2(x-60) + 95$ is a model that yields the speeding fine on the Pennsylvania Turnpike.

- I would like you to travel to a town that you have never visited. When you get off the plane I want you to imagine that all signs have been made invisible to you, but everyone else can read them. Your job is to get to your hotel. You can well imagine how lost you would be. The same is true of anyone who is ignorant of mathematics as that person tries to make his or her way through nature.
- Algebra is the first stop beyond arithmetic in quantitative thinking. It helps students learn to formulate and solve problems, to identify and communicate patterns and trends. It builds skills and patterns of thought essential to accomplishing many quantitative challenges we face in everyday life (home finances, business, trade, professions). Indeed, it is fundamental to quantitative literacy.
- To be able to deal with problem situations that arise in life without algebra as a tool for reasoning would be very difficult, if not impossible. Algebra provides the language that is used in the study of more mathematics.
- Through the study of algebra you will learn the skills of abstraction and generalization that are indispensable for operating in the information age.

Why study algebra? (addressed to college students)

- Most mathematics students in college still think of mathematics as problem-solving and computation. More algebra taught well gives them a glimpse of systematic thinking, of the essence of mathematics. The skills of advanced algebra are generalizable to computing, to logic, and to higher order reasoning. It opens intellectual and not just career doors.
- Algebra is a branch of mathematics which is essential to main areas of physics. It also has applications for the recently developed areas of error-correcting codes and cryptography. It is the language of mathematics and part of our cultural heritage.
- Algebra studies the commonalities that occur in various number systems or other objects and operations on them (matrices, transformations, logical propositions). The study of algebra shows the power of mathematics to synthesize a variety of mathematical properties from different contexts.
- Abstract algebra is the study of the structure of sets equipped with formal operations. By abstracting the essential ideas of structure, we are able to see the common features of diverse situations. Rings, for example, complete the structure of arithmetic and polynomial manipulation of high school algebra.

- Linear algebra provides a means for manipulating data. It is a means to solve systems of equations. It also provides a tool for expressing and solving geometrical problems. It is a course that has connections to many mathematical and geometric ideas.
- Linear algebra is the story of linear relationships. It started off with the complete theory of systems of linear equations, but has gone far beyond that. Often something in the real world is specified by a number of parameters—numbers like x, y, z, ... Some other facts about it are expressed by linear functions of these parameters, like

  $$X = 3x + 5y - 7z \ldots$$
  $$Y = 2x - 9y + z \ldots$$

  Linear algebra studies these linear relationships. You might have a reactor vessel, with pressure, temperature, concentrations as the starting numbers x, y, z, ... and reaction products as the new ones, X, Y, Z, ... Or it might be some other very different situation. There is a rich and powerful and non-obvious theory of all this: it's called linear algebra. In addition to real-life examples, there are lots of other examples inside mathematics itself. Indeed, linear algebra is fundamental to almost all pure mathematics as well as being the most prominent type of mathematics used in its applications.
- Why study higher algebra? In order to see the power of abstract ideas, for example, groups, which one can get by abstracting from several examples (integers, permutations), feeding back to apply to number theory and combinatorics (Burnside enumeration, chemical molecules) to give insight into many other systems.

Appendix B: A Few "Big Ideas" of Algebra

Conferees' Perspectives

- Generality, representation, quantity, structure, relationships
- The use of symbols to represent ideas or numbers
- The expression of relationships by equations
- The way expressions vary as the numbers in them change
- The difference between linear and nonlinear behavior
- Representational abilities for modeling systems with variability
- Operations on systems focusing in common properties, structures, and functions
- Power of symbols to assist reasoning and to communicate
- Modeling—transition from physical to mathematical
- Reasoning about quantitative situations
- Recognizing patterns and important (basic) functions
- Reasoning with these patterns
- Communicating ideas about these patterns in multiple representations: verbal, tabular, symbolic, graphic, and in problem situations
- Linearity vs. nonlinearity
- Equivalent representations (within and among graphs, tables, symbols)

- Variable quantities (x)
- Representation: numerical, graphic, verbal, symbolic
- Operations
- Structures
- Functions and relations
- Symbol sense
- Generality
- Representational systems
- Linearity and nonlinearity
- Relationships and functions
- Structure

- Solving systems
- Abstraction
- Algorithm (proof)

- Symbol sense
- Patterns, descriptions
- Symbol manipulation
- Obtaining unknown information from known
- Multiple representation
- Underlying structure

Appendices

Appendix A

UNITED STATES DEPARTMENT OF EDUCATION

OFFICE OF THE ASSISTANT SECRETARY
FOR EDUCATIONAL RESEARCH AND IMPROVEMENT

THE ALGEBRA INITIATIVE COLLOQUIUM
The Xerox Document University, Leesburg Virginia
December 9 through 12, 1993

Thursday evening, December 9, 1993

3:00 - 6:30 Registration

6:30 - 7:00 Cash Bar (Dining Room, Center Section)

7:00 - 9:00 Tracing the Development of Algebra and of Algebra Education in the Schools

Dinner Session

Welcome & Introduction: Joseph Conaty, Director, Office of Research, OERI

Speaker: Victor Katz, University of the District of Columbia

Discussion

Charge to Participants: Carole Lacampagne, Office of Research, OERI

9:00-9:30 Meeting of Working Group Chairs and At-Large Participants

Friday, December 10, 1993

7:00 - 8:00 Breakfast

8:30 - 10:00 Creating an Appropriate Algebra Experience for all K-12 Students (Room 3475 Red)

Speaker: James Kaput, University of Massachusetts at Dartmouth

Responders: Gail Burrill, Whitnall High School, Greenfield, Wisconsin

James Fey, University of Maryland

Discussion

10:00 - 10:30 Break

10:30 - 12:00 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

12:00 - 1:00 Lunch

1:30 - 3:00 Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer
(Room 3475 Red)

Speaker: Michael Artin, Massachusetts Institute of Technology

Responder: Vera Pless, University of Illinois at Chicago

Discussion

3:00 - 3:30 Break

3:30 - 5:00 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

6:00 - 7:00 Dinner

7:00 - 8:30 Cross-cutting Panel and Discussion
(Room 3475 Red)

Two representatives from each Working Group will outline issues discussed in their Working Groups that cut across Working Group boundaries. Whole group discussion will follow.

8:30 - 9:00 Meeting of Working Group Chairs and At-Large Participants

Saturday, December 11, 1993

8:00 - 8:45 Breakfast

8:45 - 10:15 Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce
(Room 3475 Red)

Speaker: Henry Pollak, Teachers College, Columbia University

Responder: Solomon Garfunkel, COMAP, Inc.

Discussion

10:15 - 10:45 Break

10:45 - 11:45 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

11:45 - 12:30 Lunch

1:00 - 2:30 Educating Teachers, Including K-8 Teachers, to Provide Appropriate Algebra Experiences for Their Students
(Room 3475 Red)

Speaker: Alba Gonzalez Thompson, San Diego State University

Responder: Marjorie Enneking, Portland State University

Discussion

2:30 - 3:00 Break

3:00 - 5:00 Working Group Sessions

Participants will meet with their assigned working groups for discussion of key issues

(Working Group 1, Room 3263 Yellow)
(Working Group 2, Room 3461 Red)
(Working Group 3, Room 3465 Red)
(Working Group 4, Room 3463 Red)

5:30 - 6:30 Dinner

7:00 - 8:30 Cross-cutting Groups

Representatives from each Working Group will meet in four Cross-cutting Groups to outline issues discussed in their respective Working Groups that cut across Working Group boundaries.

(Cross-cutting Group 1, Room 3263 Yellow)
(Cross-cutting Group 2, Room 3461 Red)
(Cross-cutting Group 3, Room 3465 Red)
(Cross-cutting Group 4, Room 3463 Red)

8:30 - 9:00 Meeting of Working Group Chairs and At-Large Participants

Sunday, December 12, 1993

8:00 - 8:45 Breakfast

8:45 - 9:45 Cross-cutting Working Groups Panel and Discussion (Room 3475 Red)

Two representatives from each Cross-cutting Working Group will present the conclusions of those working groups. Whole group discussion will follow.

9:45 - 10:15 Break

10:15 - 11:45 Wrap up and Look to the Future (Room 3475 Red)

Speaker: Lynn Steen, Mathematical Sciences Education Board

Discussion

Where Do We Go From Here?

Discussant: Carole Lacampagne

11:45 - 12:30 Lunch

1:00 Bus departs for Dulles Airport from Plaza B

Please leave your room key at the registration desk or at the key drop box provided at Plaza B.

Appendix B
Conceptual Framework for the Algebra Initiative of the National Institute on Student Achievement, Curriculum, and Assessment

Carole B. Lacampagne

Algebra is the language of mathematics. It opens doors to more advanced mathematical topics for those who master basic algebraic concepts. It closes doors to college and to technology-based careers for those who do not. Those most seriously affected by lack of algebraic skills are students from minority groups. Moreover, advances in the field of algebra, technology, in the needs of the work force, and in research on the teaching and learning of algebra should profoundly affect what algebra is taught and how it is taught and learned. Unfortunately, changes in the algebra curriculum and in its teaching and learning do not manifest themselves frequently in the classrooms of America. The proposed National Institute on Student Achievement, Curriculum, and Assessment initiative will confront this current crisis in the learning and teaching of algebra at all levels, kindergarten through graduate school.

Problem and Significance of the Initiative

Algebra is central to continued learning in mathematics. The position of the National Council of Teachers of Mathematics and of the ensuing reform in mathematics recognizes the need for restructuring algebra to make it part of the curriculum for all students. Moreover, reform is just under way in college curriculum and teaching of mathematics, including algebra. However, there is no coordinating mechanism to link all the groups concerned with the content, teaching and learning, and research in algebra. It is the aim of the proposed Algebra Initiative to provide this coordinating mechanism.

Conceptual Framework

Four key constructs provide a conceptual framework for the National Institute's Algebra Initiative:

- Creating an appropriate algebra experience for all grades K-12 students;

- Educating teachers, including K-8 teachers, to provide these algebra experiences;
- Reshaping algebra to serve the evolving needs of the technical work force; and
- Renewing algebra at the college level to serve the future mathematician, scientist, and engineer.

These constructs will be addressed in light of current and proposed advances in computer technology that should seriously affect what algebra is learned in the 21st century, and how it is taught.

Moreover, the uniqueness of the National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative is that these constructs will be dealt with in toto; that is, the continuum of and branching of algebra across the gamut of potential users—those whose formal experience with algebra ends in high school through the research mathematician.

Algebra for All Students

In its *Curriculum and Evaluation Standards for School Mathematics*, the National Council of Teachers of Mathematics recommends algebra for all students, beginning in the middle school years as a bridge between the concrete mathematics curriculum of the elementary school and the more formal curriculum of the high school. Moreover, algebra is to be part of a core curriculum in high school mathematics, to be taken by all students.

The National Center for Research in Mathematical Sciences Education's (NCRMSE) Working Group on Learning/Teaching of Algebra and Quantitative Analysis is also re-examining the place of algebra in a core quantitative mathematics curriculum. They are looking to reform the algebra curriculum along such coherent mathematical ideas as function and structure and to study the effect of such a curriculum on how teachers teach algebra. They are looking at the reformed curriculum in light of advances in computer software and technology which allow one to do many of the algorithmic and graphing processes of algebra on a computer or sophisticated calculator.

Besides supporting the NCRMSE Working Group on Learning/Teaching of Algebra and Quantitative Analysis, the U.S. Department of Education supports other algebra initiatives including Robert Moses' Algebra Project, a pre-algebra project designed to bridge the gap between arithmetic and algebra for minority students.

The National Science Foundation is funding a variety of projects to develop instructional materials in mathematics, K-12. Many of these materials include algebra strands. One such materials development project addresses directly a core high school mathematics curriculum which includes algebra. Several others have algebra strands cutting across several grade levels. Still others stress algebraic applications to other disciplines such as science and business.

Questions that need to be addressed include:

- What algebra concepts should be part of the core mathematics curriculum for all students?
- How should they be taught and learned?
- What is the role of technology in this curriculum and pedagogy?
- How will these algebra concepts be tied into the needs of an educated citizen and a technical worker?
- How will they be related to the continuum of algebra in college and beyond?
- How will they prepare students for lifelong learning in mathematics?

Algebra for Future School Teachers of Mathematics

With the teaching and learning of algebra moved down to the middle school years, elementary school teachers now need a thorough understanding of the underlying concepts of algebra, of how children understand/misunderstand basic algebra concepts, and of pedagogical approaches suitable to helping children develop early algebra concepts. High school teachers of mathematics need a new and different knowledge of algebra and appropriate pedagogy. They need to understand the links between algebra and other fields of mathematics as well as between algebra and its applications to the social, natural, and physical sciences.

Thus, mathematics curriculum and pedagogy for future teachers of mathematics need to be rethought in light of the changing role of algebra in the school curriculum. The Mathematical Association of America recognizes the need for a different experience in algebra for future school teachers of mathematics in its document *A Call for Change: Recommendations for the Mathematical Preparation of Teachers*. However, much work is needed to effect change in the algebraic education of future teachers.

Questions that need to be addressed include:

- What algebra concepts and pedagogy should be fundamental to mathematics courses required of future elementary school teachers? middle school teachers? high school teachers?

- How can future teachers best acquire such knowledge and skills?
- How can we enhance the algebraic and pedagogical skills of current teachers?

Algebra for the Technical Work Force

The majority of U.S. students are leaving school ill-prepared in mathematical problem solving, planning and optimizing, and mathematical modeling—mathematical tools needed in forward-thinking business and industry. All of the above areas are based on the language and operations of algebra and all should play an important role in curricula designed for those entering the technical work force—be they products of high school vocational education, 2-plus-2 plans, or 4-year college technical education curricula. Emphasis here will be on developing curricula and pedagogy to equip students to meet the demands of the 21st century work force and to provide the base for lifelong learning in mathematics as technology shifts and careers change.

Questions to be addressed include:

- What algebra concepts, beyond the core, do students preparing to enter the technical work force need?
- How should these concepts be taught and learned?
- What is the role of technology in such a curriculum and pedagogy?
- How will these algebra concepts prepare students for continuing learning in mathematics?
- Can we circumvent the dreary sequence of remedial courses in arithmetic and algebra that many community college students must take?

Algebra for the Future Mathematician, Scientist, and Engineer

Algebra is the language of mathematics. Just as school algebra allows students to communicate mathematically, to model problems and to solve them, its extensions, linear and abstract algebra form a basis for many areas of mathematics: group theory, ring theory, algebraic topology, and algebraic number theory, to name a few. Basic knowledge of linear and abstract algebra concepts are required of all college mathematics majors, and a deeper understanding of these concepts are required of all serious users of mathematics.

Recent developments in computer software have made complex matrix manipulation, graphing, and object manipulation quicker and have enabled further developments in several areas of algebra.

Key questions to be considered include:

- How should the transition from school algebra to the advanced algebra concepts taught in college be accomplished?
- How can we help students develop mathematical maturity and smooth their transition into a first proof course?
- What advanced algebra concepts should form the core of the collegiate mathematics major? for the graduate degree in mathematics?
- How should these concepts be taught and learned?
- What is the role of technology in such a curriculum and pedagogy?
- What research opportunities in algebra can be afforded students in their undergraduate and early graduate experience in mathematics?
- How will these algebra concepts prepare students for employment outside of academia?

Action Plan

The action plan for the National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative includes:

- A 3 1/2 day invitational colloquium;
- A widely distributed summary document for policymakers and teachers on the major issues discussed at the colloquium;
- A complete *Proceedings* of the colloquium to be distributed to the mathematics and mathematics education communities;
- A 2-year discretionary grant solicitation for a Linking Algebra project; and
- The linking of other initiatives, both within the Department of Education and across other federal agencies, to what is learned from the Algebra Initiative.

Invitational Colloquium

The National Institute on Student Achievement, Curriculum, and Assessment's Algebra Initiative will begin with a 3 1/2 day invitational colloquium on algebra involving about 40 key players from NCRMSE, the Algebra Project, principal investigators from NSF's algebra materials development and research projects, college teachers of algebra, and mathematics and mathematics education researchers plus additional mathematics experts from federal agencies. Carole Lacampagne will run the colloquium in coordination with William Blair, James Kaput, and Richard Lesh.

Summary Document

A report writer with experience in writing articles about mathematics will be hired to write a short document for policymakers and teachers based on issues raised at the colloquium. This document will be distributed widely within 4 months of the colloquium.

Proceedings

All speakers, discussants, and participants in the colloquium will be asked to submit papers to be published in a *Proceedings*. This document will be distributed to the mathematics community.

Further dissemination of knowledge gained through the colloquium will occur at workshops, minicourses, and talks on algebra given at annual meetings of the National Council of Teachers of Mathematics, the Society for Industrial and Applied Mathematics, and at semiannual joint meetings of the Mathematical Association of America, and the American Mathematical Society.

Two-year Discretionary Grant Solicitation for a Linking Algebra Project

It is anticipated that the Department of Education, perhaps in collaboration with other federal agencies, will fund a Linking Algebra project organized around the four key constructs of the colloquium to extend the concepts of, embark on research in, and prepare for implementation in schools and colleges knowledge gained in the colloquium.

It is anticipated that the Algebra Initiative through taking the broad view of algebra, from the teaching and learning of algebra in the elementary school through breakthroughs in research, will spur teachers of mathematics, teachers of teachers of mathematics, and the mathematics and mathematics education communities to coordinate and extend the role of algebra for all learners and users of mathematics.

Appendix C
Participant List

Working Group 1: Creating an Appropriate Algebra Experience For All K-12 Students

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Working Group 2: Educating Teachers, Including K-8 Teachers, to Provide These Algebra Experiences

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Working Group 3: Reshaping Algebra to Serve the Evolving Needs of the Technical Workforce

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Working Group 4: Renewing Algebra at the College Level to Serve the Future Mathematician, Scientist, and Engineer

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